

Copula Geometry and Second-Order Calibration of Heavily Right Aggregation

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Abstract

Heavy-tailed p -value combination tests are attractive under unknown dependence because their null tails can be first-order robust even when the exact dependent null distribution is unavailable. That robustness does not resolve the calibration problem: a statement of the form $\Pr\{T > q(\alpha)\} = \alpha + o(\alpha)$ neither quantifies the remaining size error nor determines whether the test is conservative or anticonservative. We develop a second-order calibration theory for positive Half-Cauchy and reciprocal, or harmonic-mean, aggregation.

After exact marginal standardization, extremal dependence is represented by an index-one exponent measure on the coordinate-face lattice. Its support separates axial, full-interior, proper-face, and mixed geometries. An exact Möbius decomposition shows how the noncompact weighted half-space probes every face, while quantitative face limits determine whether the resulting correction is integrable, critically amplified, or nonintegrable. Together with common-factor inversion and a tail-to-calibration map, this gives a reusable machinery that applies to dependence structures more broadly than individual parametric copula families.

The general results include an integrable hidden-face transfer theorem, a sharp fixed-dimensional local-corner theorem with explicit higher-face control, a common-heavy-factor theorem, and size and critical-value expansions relative to independence. Gaussian, standard multivariate- t , positive Clayton, and max-linear pair-shock models exhibit distinct power, logarithmic, radial-angular, and proper-face mechanisms; the Gaussian weighted-half-space transfer is conditional on explicit face-boundary hypotheses. The resulting theory shows how dependence geometry controls the rate, coefficient, and direction of the calibration error left unresolved by first-order validity. Asymptotics use $t \rightarrow \infty$, equivalently vanishing significance levels.

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1 Statistical problem and organizing principle

1.1 The calibration problem

Let exact-uniform p -values be transformed into nonnegative scores with right tail proportional to t^{-1} and aggregated with simplex weights. The two leading examples are the reciprocal score p^{-1} , whose weighted sum is the inverse harmonic mean, and the positive Half-Cauchy score $\cot(\pi p/2)$. The former is connected to the harmonic-mean p -value of Wilson [41]; the latter is the positive Half-Cauchy combination test (HCCT) in the “heavily right” program of Liu, Meng and Pillai [33]; and the Cauchy-combination starting point is due to Liu and Xie [34].

These methods address a persistent difficulty in combining dependent evidence. The joint null law of the aggregate is generally unavailable without specifying or estimating the full dependence structure. Heavy right tails partly bypass that obstacle: under index-one multivariate regular variation (MRV), linearity and exact marginal normalization make the leading tail constant independent of the spectral measure. This projection identity is classical in MRV and now appears explicitly in the heavy-tailed-combination literature [28, 22]. It explains first-order robustness, but it does not provide exact calibration.

Indeed, let $S_{\mathbf{w}}$ denote either the reciprocal aggregate or the positive Half-Cauchy aggregate, and let $q_{\Pi, S, \mathbf{w}}(\alpha)$ be its exact upper α -quantile under independent p -values. First-order validity only asserts

$$\Pr_C\{S_{\mathbf{w}} > q_{\Pi, S, \mathbf{w}}(\alpha)\} = \alpha + o(\alpha), \quad \alpha \downarrow 0.$$

It does not determine how large the error is at a small but finite level, whether the reference is conservative or anticonservative, or how its critical value should be corrected. Those questions are precisely the second-order calibration problem. They are statistically consequential because unavailable exact dependent calibration is the central price paid for the robustness and simplicity of heavy-tailed aggregation.

1.2 Relation to existing work

The harmonic-mean p -value of Wilson [41] aggregates reciprocal scores, while the heavily right framework of Liu, Meng and Pillai [33] studies positive Half-Cauchy scores and builds on the Cauchy combination literature [34, 7]. The two nonnegative rules share their index-one leading geometry but differ at second order because reciprocal scores are exactly Pareto whereas the Half-Cauchy map has an intrinsic analytic remainder.

Recent work establishes first-order validity and angular-measure calibration for heavy-tailed combination tests under several dependence regimes [27, 28, 22]; finite-level harmonic-mean inequalities provide a complementary perspective [23]. The classical MRV, hidden-regular-variation, and coordinate-face foundations are not new [1, 6, 30, 37, 38, 20].

Second-order aggregation of heavy-tailed risks is also established theory [26, 21, 12, 11]. In particular, the unit-index logarithm and related integrability boundary appear in existing bivariate copula-sum expansions [42]. Gaussian hidden tails, Archimedean frailty, standard- t tail dependence, and max-linear spectral atoms likewise have substantial prior literatures. Accordingly, the contribution claimed here is not those ingredients separately. It is the common independence-relative calibration problem for harmonic-mean and positive Half-Cauchy aggregation, together with a face-lattice calculus that tracks the rate, coefficient, sign, and critical-value consequence across support regimes. An extended literature review and a claim-by-claim novelty audit are in [Appendix C](#).

1.3 Three contributions

The paper develops one chain of ideas:

$$\begin{aligned} \text{dependence geometry} &\implies \text{face-specific tail correction} \\ &\implies \text{aggregate-tail expansion} \implies \text{size and critical-value distortion.} \end{aligned} \tag{1.1}$$

Its contributions are best understood in three parts.

1. *Second-order calibration as statistical theory.* We take the exact independence law as the operational reference and derive the leading dependence-induced error in its tail probability, rejection probability, and critical value. The output is a rate, coefficient, and sign, rather than an unquantified $o(\alpha)$ guarantee.
2. *A reusable second-order machinery.* Exact Pareto standardization, quantitative exponent measures, an exact Möbius decomposition of the weighted half-space, hidden-face transfer, local-corner analysis, common-factor inversion, and calibration inversion form a common calculus. This machinery separates perturbations of strata already charged at first order from hidden mass on new faces, critical accumulation near subfaces, nonintegrable boundary layers, and score-transformation effects.
3. *A geometric organization of dependence.* The support of the exponent measure on the coordinate-face lattice gives four primary geometries: axes, full interior, nontrivial proper faces, and mixtures of these strata. More importantly, the way probability approaches charged and uncharged faces predicts the possible second-order mechanism. The resulting organization groups copulas and more general dependence structures by their extremal geometry rather than merely by family name.

The general theory yields a sharp local-corner theorem in arbitrary fixed dimension, with unequal weights and explicit control of higher faces; an integrable hidden-face transfer theorem; a common-heavy-factor theorem; and a calculus converting tail coefficients into size and critical-value errors. The model results are designed to expose distinct parts of the machinery. Independence is the critical axial benchmark; Gaussian copulas exhibit hidden pair and boundary phenomena; standard multivariate- t and positive Clayton copulas exhibit radial-angular interior or mixed behavior; and a max-linear pair-shock construction gives an exact proper-face example.

The geometry is summarized at three resolutions. Table 2 gives the four principal measure-transfer mechanisms, Table 3 compares the Gaussian, independence, standard- t , and positive Clayton benchmarks, and the full ledger in Table 5 in Appendix B records model-specific support, second-order rate, mechanism, proof status, and bounded-density diagnostics. The detailed ledger and its supporting calculations are placed there so that the main development can emphasize the general geometry.

1.4 Paper organization

The argument links testing, copulas, multivariate extremes, and heavy-tail aggregation. We review the needed MRV ingredients, emphasizing exponent measures, the index-one projection identity, and spectral support on the coordinate-face lattice.

Section 2 develops the score transformations and the exponent-measure geometry. The lower-tail copula formulation is treated in the following section. We then introduce the calibration functional and the face calculus before turning to the four geometric regimes. The quantitative assumption hierarchy and the generic common-factor inversion theorem are stated in Appendix A, while their consequences needed for the main argument are summarized at the relevant points. Each model analysis follows the same route: identify the first-order support, locate the finite-threshold mass that can move the weighted sum across its boundary, determine the corresponding rate and coefficient, and translate the result into calibration error for both reciprocal and positive Half-Cauchy aggregation. Statements in theorem, proposition, lemma, and corollary environments are claimed under their displayed hypotheses; conjectures, open problems, and classifications labelled *expected* are not claimed as proved. All formal proofs are collected in Appendix D.

2 Heavily right scores and exponent-measure geometry

2.1 Harmonic-mean and positive Half-Cauchy scores

Let $m \geq 1$ be fixed and let $p_1, \dots, p_m$ have exact Uniform(0, 1) null margins, with dependence unrestricted until stated otherwise. Put $U_i = p_i$, $Y_i = 1/U_i$, $H_i = \cot(\pi U_i/2)$, and $c = 2/\pi$. For simplex weights, define

$$R_{\mathbf{w}} = \sum_{i=1}^m w_i Y_i = \sum_{i=1}^m \frac{w_i}{p_i}, \quad T_{\mathbf{w}} = \sum_{i=1}^m w_i H_i = T_{\text{HCCT}}, \quad w_i \geq 0, \quad \sum_{i=1}^m w_i = 1. \quad (2.1)$$

Here $R_{\mathbf{w}}$ is the inverse weighted harmonic mean and $T_{\mathbf{w}}$ is the positive Half-Cauchy statistic. Zero-weight coordinates may be deleted. Exactly, $\mathbb{P}(Y_i > y) = y^{-1}$ for $y \geq 1$, whereas $\mathbb{P}(H_i > x) = 2 \arctan(x^{-1})/\pi = c/x + O(x^{-3})$. We write $F_{R, \mathbf{w}}$ and $F_{H, \mathbf{w}}$ for the corresponding CDFs under independent exact-uniform p_i and fixed weights.

Proposition 2.1 (Pareto–Half-Cauchy equivalence). *For $h(u) = \cot(\pi u/2)$ and $d(u) = c/u - h(u)$, the function d is increasing from 0 to c on $[0, 1]$. Hence*

$$cY_i - c \leq H_i \leq cY_i, \quad \|\mathbf{H} - c\mathbf{Y}\|_{\infty} \leq c. \quad (2.2)$$

For fixed m , $\mathbf{Y} \in \text{MRV}_1(\mu_Y)$ if and only if $\mathbf{H} \in \text{MRV}_1(\mu_H)$, with $\mu_H(A) = \mu_Y(c^{-1}A)$ and $\mu_H\{\mathbf{x} : x_i > 1\} = c$ for every i .

Thus the right object to inspect in the original p -value copula is the Pareto vector $(1/p_1, \dots, 1/p_m)$. The two score vectors therefore have the same first-order geometry up to scale, but their second-order terms can differ because only the reciprocal margins are exactly Pareto. Indeed, $|T_{\mathbf{w}} - cR_{\mathbf{w}}| \leq c$. Under the mild local tail regularity needed to control this bounded threshold displacement,

$$\begin{aligned} \mathbb{P}(R_{\mathbf{w}} > s) &= s^{-1} + Bs^{-\beta}L(s) + o\{s^{-\beta}L(s)\}, \quad s^{-\beta}L(s) \gg s^{-2}, \\ \implies \mathbb{P}(T_{\mathbf{w}} > t) &= \frac{c}{t} + Bc^{\beta}t^{-\beta}L(t/c) + o\{t^{-\beta}L(t)\}. \end{aligned} \quad (2.3)$$

This covers the positive-correlation Gaussian term, the independent $(\log t)/t^2$ term, and the multivariate- t radial term for $\nu > 2$. At order t^{-2} or smaller the bounded displacement can change the coefficient, so the two aggregates must be expanded separately.

2.2 Multivariate regular variation and its polar representation

Put

$$\mathbb{E}_m = [0, \infty]^m \setminus \{\mathbf{0}\}.$$

A set is bounded away from the origin if it does not intersect some neighborhood of $\mathbf{0}$.

Definition 2.2 (Index-one MRV). *A nonnegative random vector $\mathbf{X} = (X_1, \dots, X_m)$ is multivariate regularly varying with index one, written $\mathbf{X} \in \text{MRV}_1(\mu)$, if there exists a nonzero Radon measure μ on $\mathbb{E}_m$ such that*

$$t \mathbb{P}\left(\frac{\mathbf{X}}{t} \in A\right) \longrightarrow \mu(A) \quad (2.4)$$

for every Borel set $A \subset \mathbb{E}_m$ that is bounded away from the origin and satisfies $\mu(\partial A) = 0$.

The measure μ is called the exponent or tail measure. It is homogeneous of order -1 :

$$\mu(aA) = a^{-1}\mu(A), \quad a > 0. \quad (2.5)$$

Using L^1 polar coordinates

$$r = \|\mathbf{x}\|_1, \quad \mathbf{s} = \frac{\mathbf{x}}{\|\mathbf{x}\|_1}, \quad \mathbb{S}_+^{m-1} = \{\mathbf{s} \geq 0 : \|\mathbf{s}\|_1 = 1\},$$

the tail measure has the representation

$$\mu(dr, d\mathbf{s}) = r^{-2} dr S(d\mathbf{s}), \quad (2.6)$$

where S is a finite measure on the positive simplex. The location of S describes extremal dependence:

- mass only on the coordinate axes means asymptotic tail independence;
- mass away from the axes means that simultaneous extremes have first-order probability;
- mass on the diagonal is the limiting geometry of perfect positive tail dependence.

MRV is therefore not synonymous with tail independence. Both Gaussian copulas with correlations strictly below one and multivariate- t copulas are covered, but their spectral measures have different supports.

Exponent-measure interpretation The exponent measure is the intensity measure of the limiting Poisson cloud of normalized extremes. Its traditional name comes from the max-stable representation: exponentiating the negative of the relevant exceedance mass gives the limiting distribution. It is unrelated to a Lie-theoretic exponential. The full point-process and max-stability derivation is given in [Appendix A](#). What matters for aggregation is that the measure records both the radial frequency of large observations and the angular locations at which they occur.

2.3 Spectral measures and coordinate-face geometry

For unit-Pareto margins, the spectral measure obeys

$$\int_{\mathbb{S}_+^{m-1}} s_i S(ds) = 1, \quad i = 1, \dots, m. \quad (2.7)$$

Consequently $S(\mathbb{S}_+^{m-1}) = m$. Conversely, any finite positive measure on the simplex satisfying [Eq. \(2.7\)](#) defines an index-one exponent measure with unit-Pareto margins. Radially the measure is rigid, because [Eq. \(2.6\)](#) puts the same $r^{-2}dr$ intensity along every charged direction. Angularly it is flexible: S may be atomic, diffuse, singular, supported on lower faces, or any mixture of these, subject only to the marginal moment constraints.

For nonempty $I \subseteq [m]$, define the relative interior of the coordinate face

$$F_I^\circ = \{s \in \mathbb{S}_+^{m-1} : s_i > 0 \ (i \in I), \ s_j = 0 \ (j \notin I)\}. \quad (2.8)$$

It is useful to record the strata carrying positive spectral mass:

$$C(S) = \{I \subseteq [m] : I \neq \emptyset, \ S(F_I^\circ) > 0\}. \quad (2.9)$$

This measure-theoretic definition avoids confusing mass on a relative-open stratum with topological support in its closure. It gives four mutually exclusive first-order geometries:

1. *axial*: every $I \in C(S)$ has $|I| = 1$;
2. *interior*: $C(S) = \{[m]\}$, equivalently S assigns the simplex boundary zero mass;
3. *proper-face*: every $I \in C(S)$ satisfies $2 \leq |I| \leq m - 1$;
4. *mixed*: more than one of the preceding three stratum types carries positive mass.

The shorthand classifies the exponent measure associated with a copula, not the support of the copula distribution itself. Proper faces do not exist when $m = 2$. A ray is an atom of S at a direction and is not a fifth category: it is axial, proper-face, or interior according to the relative-open stratum containing that direction. With equal unit-Pareto marginal constants, a spectral measure supported on a single ray must use the full diagonal; several asymmetric rays can balance one another.

The four panels of [Fig. 1](#) describe where first-order extremes occur. They do not determine how fast finite-threshold probability approaches those strata. In particular, independence, Gaussian copulas with correlations below one, exclusion models, and rate-free constructions may all be axial while having fundamentally different second-order behavior. That separation between support and approach rate is the reason geometry and quantitative face analysis enter as distinct parts of the theory.

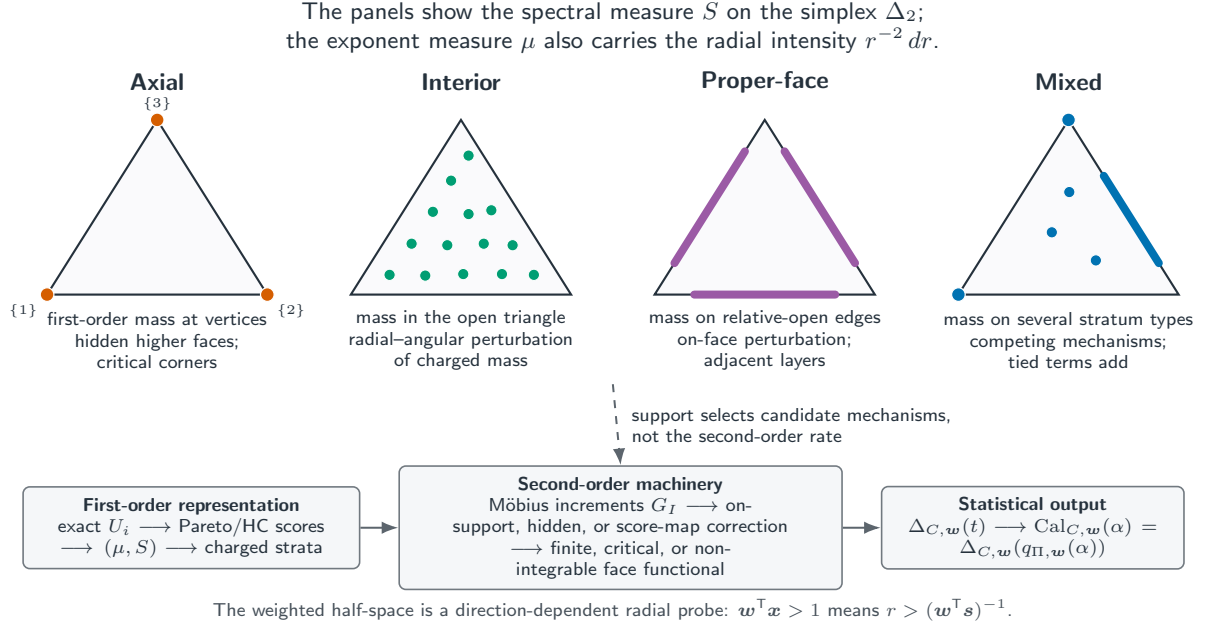


Figure 1: Exponent-measure geometry and the route to calibration in dimension three. The upper panels depict positive mass of the angular spectral measure S on relative-open simplex strata; they do not depict the radial exponent measure μ itself. An interior measure may approach the boundary in topological support while assigning it zero mass, and atoms or rays may occur within any charged stratum. The lower strip shows the paper’s organizing chain. First-order support selects candidate second-order mechanisms, whereas quantitative convergence and integration against the signed weighted-half-space increments determine the rate and coefficient. The final target is error relative to exact independence calibration, rather than the raw tail’s second term.

2.4 Tail index one and universal first-order aggregation

For a general tail index $\alpha > 0$, MRV is formulated with a scaling $b(t) \in \text{RV}_{1/\alpha}$:

$$t\mathbb{P}\{X/b(t) \in \cdot\} \xrightarrow{v} \mu(\cdot), \quad \frac{b(tx)}{b(t)} \longrightarrow x^{1/\alpha}. \quad (2.10)$$

The exponent measure is homogeneous: $\mu(aA) = a^{-\alpha} \mu(A)$.

Its polar form is $\mu(dr, ds) = \alpha r^{-\alpha-1} dr S(ds)$. A Pareto- α tail $\mathbb{P}(X > x) = x^{-\alpha}$ has maxima of order $n^{1/\alpha}$. Smaller α means a heavier tail.

The reciprocal margins have index one exactly, while [Proposition 2.1](#) transfers the same index to the positive Half-Cauchy scores. This statement concerns the transformed p -value scores: a raw standard multivariate- t_ν vector has index ν , and a raw Gaussian vector is not regularly varying, even though their exact- p reciprocal scores have index one.

Marginal Pareto standardization can convert continuous margins to index one, but it changes aggregation on the original scale. Index one is especially consequential here. For a general index α , let $A_w = \{x \in \mathbb{E}_m : w^\top x > 1\}$. Radial integration gives

$$\mu(A_w) = \int (w^\top s)^\alpha S(ds), \quad (2.11)$$

where the marginal tail constants are $c_i = \int s_i^\alpha S(ds)$. Only at $\alpha = 1$ does the projection linearize:

$$\mu(A_{\mathbf{w}}) = \int \mathbf{w}^\top \mathbf{s} S(ds) = \sum_{i=1}^m w_i c_i. \quad (2.12)$$

This is why the leading constants below depend on the margins and weights, but not on whether the spectral measure is axial, interior, proper-face, or mixed.

Theorem 2.3 (Universal first-order harmonic-mean and Half-Cauchy tails). *Let m be fixed. Suppose that the p_i have exact uniform margins, that $\mathbf{w}$ is a simplex weight vector, and that $\mathbf{Y} = (1/p_1, \dots, 1/p_m)$ is multivariate regularly varying with index one. Equivalently, by [Proposition 2.1](#), $\mathbf{H}$ is multivariate regularly varying with index one. Then*

$$\mathbb{P}(R_{\mathbf{w}} > t) = \frac{1}{t} + o(t^{-1}), \quad (2.13)$$

$$\mathbb{P}(T_{\mathbf{w}} > t) = \frac{2}{\pi t} + o(t^{-1}). \quad (2.14)$$

The same leading terms hold under independent exact-uniform p -values; therefore

$$\frac{\mathbb{P}(R_{\mathbf{w}} > t)}{1 - F_{R,\mathbf{w}}(t)} \longrightarrow 1, \quad \frac{\mathbb{P}(T_{\mathbf{w}} > t)}{1 - F_{\text{HC},\mathbf{w}}(t)} \longrightarrow 1. \quad (2.15)$$

The corresponding first-order thresholds are $t_{R,\alpha} = \frac{1}{\alpha}$ and $t_{T,\alpha} = \frac{2}{\pi\alpha}$. Thus [Theorem 2.3](#) gives

$$\mathbb{P}(R_{\mathbf{w}} > t_{R,\alpha}) = \alpha + o(\alpha), \quad \mathbb{P}(T_{\mathbf{w}} > t_{T,\alpha}) = \alpha + o(\alpha). \quad (2.16)$$

For the Half-Cauchy statistic one may instead use the marginal quantile $\cot(\pi\alpha/2)$, which is asymptotic to $2/(\pi\alpha)$. These are $\alpha \downarrow 0$ statements, not claims of exact calibration at a fixed positive level. The common first-order limit also does not say that the scores are independent; it uses nonnegativity, index one, exact marginal normalization, and simplex weights.

3 Lower max-domains for p-value copulas

3.1 Lower extremes and stable-tail dependence

Definition 3.1 (Lower max-domain of attraction). *The copula of $\mathbf{U}$ is in the lower max-domain of attraction if, for every $\mathbf{x} = (x_1, \dots, x_m) \in [0, \infty)^m$, the following limit exists:*

$$\ell_L(\mathbf{x}) = \lim_{s \downarrow 0} \frac{1}{s} \mathbb{P}\left(\bigcup_{i=1}^m \{U_i \leq sx_i\}\right). \quad (3.1)$$

Exact uniform margins give, for every $s > 0$ small enough,

$$s \max_i x_i \leq \mathbb{P}\left(\bigcup_i \{U_i \leq sx_i\}\right) \leq s \sum_i x_i. \quad (3.2)$$

Thus the union probability is automatically of order s ; lower MDA asks whether its coefficient settles to a limit. The resulting function ℓ_L is convex and homogeneous of degree one and satisfies

$$\max_i x_i \leq \ell_L(\mathbf{x}) \leq \sum_{i=1}^m x_i. \quad (3.3)$$

For $m = 2$, define the lower-tail copula, when the limit exists, by

$$\Lambda_L(x, y) = \lim_{s \downarrow 0} \frac{C(sx, sy)}{s}.$$

Inclusion–exclusion gives

$$\ell_L(x, y) = x + y - \Lambda_L(x, y). \quad (3.4)$$

For the reciprocal vector $\mathbf{Y} = 1/\mathbf{U}$, [Theorem 3.2](#) turns lower MDA into index-one MRV. If S_Y is its spectral measure, the lower stable-tail function, exponent function, and spectral representation are the same object in reciprocal coordinates:

$$\ell_L(\mathbf{x}) = V_Y(1/x_1, \dots, 1/x_m) = \int_{\mathbb{S}_+^{m-1}} \max_i(x_i s_i) S_Y(ds), \quad (3.5)$$

with the usual convention for zero coordinates.

In two dimensions, write $\mathbf{s} = (r, 1 - r)$. Then $S_Y/2$ is a probability distribution on $[0, 1]$ with mean $1/2$. Independence has $S_Y = \delta_0 + \delta_1$, perfect dependence has $S_Y = 2\delta_{1/2}$, and

$$S_Y = (1 - \lambda)(\delta_0 + \delta_1) + 2\lambda\delta_{1/2} \quad (3.6)$$

interpolates between them. The lower-tail copula has the spectral form

$$\Lambda_L(x, y) = \int_0^1 \min\{xr, y(1 - r)\} S_Y(dr). \quad (3.7)$$

Consequently, positive interior spectral mass is equivalent to $\Lambda_L(1, 1) > 0$. In that case,

$$\mathbb{P}(Y_1 > t, Y_2 > t) \sim \frac{\Lambda_L(1, 1)}{t}. \quad (3.8)$$

Axis support makes this constant zero and gives only $o(t^{-1})$; it does not determine the smaller rate.

The ratio $C(sx, sy)/(sx)$ converges to $\Lambda_L(x, y)/x$. Thus zero means simultaneous order- s events are negligible relative to a single small p -value, while a positive limit means that joint extremes already occur at first order. The union/intersection identities and their order-statistic interpretation are given in [Appendix A](#).

For $m > 2$, pairwise lower-tail limits do not generally determine the full multivariate lower-MDA limit, because higher-order intersections may contain additional information. A useful sufficient condition for axis-supported convergence is stated in [Proposition 3.3](#).

For independent replicates, normalized componentwise minima of $\mathbf{U}$ converge to a min-stable law with exponent ℓ_L ; equivalently, normalized componentwise maxima of $\mathbf{Y} = 1/\mathbf{U}$ converge to the reciprocal max-stable law. This is the origin of the phrase lower max-domain of attraction. The full replicate calculation is in [Appendix A](#).

Hidden regular variation, residual tail orders, and higher-order tail densities describe the smaller probability of simultaneous extremes when the first-order exponent measure is axial [[30](#), [31](#), [37](#), [38](#), [29](#), [32](#)]. For Gaussian copulas, reciprocal and positive Half-Cauchy scores have axial first-order MRV, while smaller-order joint tails set the calibration rate.

Lower MDA need not imply tail independence. Its universal first-order aggregation consequence is stated in [Corollary 3.4](#), while the geometry remains decisive at second order.

3.2 From lower MDA to MRV and universal aggregation

Theorem 3.2 (Copula criterion for index-one MRV). *If the limit in Eq. (3.1) exists for every $\mathbf{x} \geq 0$, then $\mathbf{Y} = (1/U_1, \dots, 1/U_m)$ is multivariate regularly varying with index one. Consequently, $\mathbf{H}$ is multivariate regularly varying with index one.*

Remark (How close this condition is to necessary). For a fixed finite dimension and exact uniform margins, existence of the limits in Eq. (3.1) is one of the standard equivalent formulations of index-one MRV of $(1/U_1, \dots, 1/U_m)$. It is therefore not merely an ad hoc sufficient condition; it is the natural copula formulation of the desired tail property.

Proposition 3.3 (An easier condition implying axis-supported MRV). *Suppose that, for every $i \neq j$ and every fixed $x, y > 0$,*

$$\mathbb{P}(U_i \leq sx, U_j \leq sy) = o(s), \quad s \downarrow 0. \quad (3.9)$$

Then

$$\ell_L(\mathbf{x}) = \sum_{i=1}^m x_i, \quad (3.10)$$

so $\mathbf{Y}$ and $\mathbf{H}$ are MRV with spectral mass only on the coordinate axes.

Condition Eq. (3.9) is convenient but substantially stronger than general MRV. It excludes any copula with nonzero lower tail dependence.

The one-big-jump analysis of Liu, Meng and Pillai [33] uses a stronger asymmetric pairwise condition. In the lower-lower form relevant to the two nonnegative score transformations studied here, there is a sequence $\delta_t \downarrow 0$ with $t\delta_t \rightarrow \infty$ such that, for every ordered pair of active coordinates $i \neq j$,

$$\mathbb{P}\left(U_i < \frac{a_i}{t}, U_j < \frac{a_j}{\delta_t t}\right) = o(t^{-1}), \quad a_k = \frac{2mw_k}{\pi} > 0. \quad (3.11)$$

This condition implies Eq. (3.9). Indeed, for fixed $x, y > 0$, put $T = a_i/(sx)$. The first threshold in Eq. (3.11) then equals sx , whereas the second equals

$$s \frac{a_j x}{a_i \delta_T},$$

which eventually exceeds sy because $\delta_T \rightarrow 0$. Therefore

$$\mathbb{P}(U_i \leq sx, U_j \leq sy) \leq \mathbb{P}\left(U_i < \frac{a_i}{T}, U_j < \frac{a_j}{\delta_T T}\right) = o(T^{-1}) = o(s).$$

Thus the earlier condition is a sufficient route to the axis-supported case of Proposition 3.3; it is not necessary for universal first-order aggregation under lower MDA.

For a decisive contrast, if $U_1 = \dots = U_m = U$, simplex weighting gives

$$R_w = U^{-1}, \quad T_w = \cot(\pi U/2).$$

The pairwise condition fails maximally, but both aggregates reduce exactly to their one-dimensional reference scores. General lower MDA therefore covers first-order extremal dependence that the earlier one-big-jump condition deliberately excludes.

Corollary 3.4 (Universal first-order tails under lower MDA). *Let m be fixed, let the p_i have exact uniform margins, and let $\mathbf{w}$ be a simplex weight vector. If the copula of $\mathbf{U}$ is in the lower max-domain of attraction in the sense of [Definition 3.1](#), then*

$$\mathbb{P}(R_{\mathbf{w}} > t) = \frac{1}{t} + o(t^{-1}), \quad (3.12)$$

$$\mathbb{P}(T_{\mathbf{w}} > t) = \frac{2}{\pi t} + o(t^{-1}). \quad (3.13)$$

Each tail is asymptotically equivalent to its exact independence counterpart.

The lower-MDA assumption is substantive Exact uniform margins alone do not force the limit in [Eq. \(3.1\)](#) to exist. [Appendix A](#) gives a scale-oscillatory copula diagonal for which the normalized lower-tail overlap has two subsequential limits. The example separates failure of a full tail measure from failure of one particular aggregate-tail limit.

4 Second-order calibration and quantitative tail control

4.1 From tail expansions to second-order calibration

The word *calibration* will refer to comparison with the exact null law under independent p -values, not merely comparison with the leading index-one tail c_0/t . This distinction matters because the independent law itself has a critical logarithmic correction for both aggregates.

Let $S_{C,\mathbf{w}}$ be a fixed weighted aggregation statistic under copula C , and write

$$\bar{F}_{C,\mathbf{w}}(t) = \mathbb{P}_C(S_{C,\mathbf{w}} > t).$$

Let Π denote the independence copula and define

$$D_{C,\mathbf{w}}(t) = \bar{F}_{C,\mathbf{w}}(t) - \bar{F}_{\Pi,\mathbf{w}}(t), \quad (4.1)$$

$$R_{C,\mathbf{w}}(t) = \frac{\bar{F}_{C,\mathbf{w}}(t)}{\bar{F}_{\Pi,\mathbf{w}}(t)} - 1. \quad (4.2)$$

For a nominal level α , let $q_{\Pi,\mathbf{w}}(\alpha)$ be the upper α -quantile of the independence law. Whenever $\bar{F}_{\Pi,\mathbf{w}}\{q_{\Pi,\mathbf{w}}(\alpha)\} = \alpha$, the actual size distortion is

$$\text{Cal}_{C,\mathbf{w}}(\alpha) = \mathbb{P}_C\{S_{C,\mathbf{w}} > q_{\Pi,\mathbf{w}}(\alpha)\} - \alpha = D_{C,\mathbf{w}}\{q_{\Pi,\mathbf{w}}(\alpha)\}. \quad (4.3)$$

Positive and negative signs indicate anticonservative and conservative independence calibration, respectively.

Theorem 4.1 (Second-order calibration and quantile inversion). *Let $j \in \{C, \Pi\}$, let $c_0 > 0$, and suppose that the two survival functions are eventually continuous and strictly decreasing and satisfy*

$$\bar{F}_{j,\mathbf{w}}(t) = \frac{c_0}{t} + d_{j,\mathbf{w}} t^{-1-\kappa} L(t) + o\{t^{-1-\kappa} L(t)\}, \quad \kappa > 0, \quad (4.4)$$

where L is positive and slowly varying. Put $q_0(\alpha) = c_0/\alpha$. Then

$$q_{j,\mathbf{w}}(\alpha) = q_0(\alpha) + \frac{d_{j,\mathbf{w}}}{c_0} q_0(\alpha)^{1-\kappa} L\{q_0(\alpha)\} + o\left(q_0(\alpha)^{1-\kappa} L\{q_0(\alpha)\}\right), \quad (4.5)$$

$$\text{Cal}_{C,\mathbf{w}}(\alpha) = (d_{C,\mathbf{w}} - d_{\Pi,\mathbf{w}}) c_0^{-1-\kappa} \alpha^{1+\kappa} L(c_0/\alpha) + o\{\alpha^{1+\kappa} L(1/\alpha)\}, \quad (4.6)$$

$$\frac{\text{Cal}_{C,\mathbf{w}}(\alpha)}{\alpha} = (d_{C,\mathbf{w}} - d_{\Pi,\mathbf{w}}) c_0^{-1-\kappa} \alpha^{\kappa} L(c_0/\alpha) + o\{\alpha^{\kappa} L(1/\alpha)\}. \quad (4.7)$$

Moreover, the dependence-specific correction to the independence critical value is

$$q_{C,\mathbf{w}}(\alpha) - q_{\Pi,\mathbf{w}}(\alpha) = \frac{d_{C,\mathbf{w}} - d_{\Pi,\mathbf{w}}}{c_0} q_0(\alpha)^{1-\kappa} L\{q_0(\alpha)\} + o\left(q_0(\alpha)^{1-\kappa} L\{q_0(\alpha)\}\right). \quad (4.8)$$

Remark (Different competing rates). If the dependent and independent corrections have different orders, first subtract the two survival expansions and retain the slower term. In particular, if

$$D_{C,\mathbf{w}}(t) = d_{\Delta,\mathbf{w}} t^{-1-\kappa} L(t) + o\{t^{-1-\kappa} L(t)\},$$

then Eq. (4.3) and $q_{\Pi,\mathbf{w}}(\alpha) \sim c_0/\alpha$ directly give

$$\text{Cal}_{C,\mathbf{w}}(\alpha) = d_{\Delta,\mathbf{w}} c_0^{-1-\kappa} \alpha^{1+\kappa} L(c_0/\alpha) + o\{\alpha^{1+\kappa} L(1/\alpha)\}.$$

This is the form used when, for example, under the face-transfer hypotheses a positive-correlation Gaussian power correction dominates the independent logarithm.

A hierarchy of quantitative assumptions Ordinary lower MDA supplies no rate. A second-order tail-measure bound on the weighted half-space does, while shell-probability, score-density, and bounded-copula-density assumptions are progressively stronger sufficient conditions near independence. The precise one-way implications and the fact that none is necessary are recorded in [Appendix A](#).

4.2 Critical examples

Two boundary examples explain why a numerical second-order rate is extra structure and why even the product copula sits at a nontrivial critical boundary.

Ordinary MRV can converge more slowly than every positive power. An exact-uniform axial example with a nonregular, arbitrarily slow diagonal remainder is constructed in [Appendix A](#); hence no second-order rate follows from first-order support alone.

Theorem 4.2 (Product copula and the critical logarithm). *Let $Y_1, \dots, Y_m$ be independent standard Pareto variables, $\mathbb{P}(Y_i > y) = y^{-1}$ for $y \geq 1$, and let $\mathbf{w}$ be a fixed simplex weight vector. Then*

$$\mathbb{P}\left(\sum_{i=1}^m w_i Y_i > t\right) = \frac{1}{t} + 2 \sum_{i < j} w_i w_j \frac{\log t}{t^2} + O(t^{-2}). \quad (4.9)$$

For $m = 2$ and $w_1, w_2 > 0$, the exact identity, valid for $t > w_1 + w_2$, is

$$\mathbb{P}(w_1 Y_1 + w_2 Y_2 > t) = \frac{w_1 + w_2}{t} + \frac{w_1 w_2}{t^2} \log \frac{(t - w_1)(t - w_2)}{w_1 w_2}. \quad (4.10)$$

At a zero-weight endpoint, this reduces to the corresponding one-dimensional Pareto identity. If H_i are independent standard positive Half-Cauchy variables and $c = 2/\pi$, then

$$\mathbb{P}\left(\sum_{i=1}^m w_i H_i > t\right) = \frac{c}{t} + 2c^2 \sum_{i < j} w_i w_j \frac{\log t}{t^2} + O(t^{-2}). \quad (4.11)$$

Thus the logarithmic rate cannot generally be improved to $O(t^{-2})$.

The product example reveals the main geometric subtlety. Rectangle probabilities have the exact scale $C_{\Pi}(sx, sy) = s^2 xy$, yet the noncompact weighted half-space reaches both axes. The hidden product measure is only critically integrable there, and the Pareto-proxy support cutoff at distance $1/t$ creates the additional $\log t$. Rectangle rates alone therefore do not determine a sum-tail rate without boundary control.

Concrete near-independence criteria Joint shell bounds imply the aggregate rate $c_X/t + O\{(\log t)/t^2\}$ for both reciprocal and positive Half-Cauchy scores. Product-type pair densities imply those shell bounds, and bounded lower-tail copula densities are a still stronger sufficient condition. Formal statements and higher-dimensional qualifications appear in [Appendix A](#).

Quantitative half-space projection If the exponent-measure approximation on $A_{\mathbf{w}} = \{\mathbf{x} : \mathbf{w}^\top \mathbf{x} > 1\}$ has relative error $O\{a(t)\}$, the aggregate has absolute tail error $O\{a(t)/t\}$. This elementary projection is useful only when the noncompact half-space is an admissible continuity set; rectangle rates alone do not provide the required boundary control. Formal details are in [Appendix A](#).

5 Face calculus for higher-order aggregation

The first-order exponent measure may live on axes, on higher-dimensional faces, or in the full orthant. A second-order sum expansion must therefore identify which face first changes the weighted half-space probability. The following exact decomposition is the bookkeeping device used below. Möbius inversion for multivariate excess and face measures is part of the existing geometric extreme-value toolkit; see Balkema and Embrechts [20]. The particular object needed here is the Möbius increment of a weighted half-space. For a nonempty set $I \subseteq [m]$, put

$$g_I(\mathbf{x}_I) = \mathbf{1}\left\{\sum_{i \in I} w_i x_i > 1\right\}, \quad g_\emptyset = 0,$$

and define its face increment by Möbius inversion,

$$G_I(\mathbf{x}_I) = \sum_{J \subseteq I} (-1)^{|I|-|J|} g_J(\mathbf{x}_J). \quad (5.1)$$

Lemma 5.1 (Exact face decomposition). *For every $\mathbf{x} \in [0, \infty)^m$,*

$$\mathbf{1}\{\mathbf{w}^\top \mathbf{x} > 1\} = \sum_{\emptyset \neq I \subseteq [m]} G_I(\mathbf{x}_I). \quad (5.2)$$

Consequently, the tail of a weighted sum is the sum of its one-coordinate contributions and of the interaction increments on all higher faces.

Vague convergence on an open face does not by itself imply [Eq. \(5.5\)](#): G_I is neither compactly supported nor bounded away from proper subfaces. The next theorem gives the needed condition.

Theorem 5.2 (Integrable hidden-face transfer). *Let $I \subseteq [m]$, $|I| \geq 2$, and let $r_I(t) \downarrow 0$. Suppose*

$$r_I(t)^{-1} \mathbb{P}(\mathbf{X}_I/t \in \cdot) \xrightarrow{v} \nu_I(\cdot) \quad \text{on } (0, \infty]^I. \quad (5.3)$$

Assume that the discontinuity set of G_I is ν_I -null, $\int |G_I| d\nu_I < \infty$, and that for

$$K_{\varepsilon, M} = \{\mathbf{x}_I : \varepsilon \leq x_i \leq M, i \in I\}$$

one has

$$\lim_{\varepsilon \downarrow 0, M \uparrow \infty} \limsup_{t \rightarrow \infty} r_I(t)^{-1} \mathbb{E}[|G_I(\mathbf{X}_I/t)| \mathbf{1}\{\mathbf{X}_I/t \notin K_{\varepsilon, M}\}] = 0. \quad (5.4)$$

Then

$$\mathbb{E}G_I(\mathbf{X}_I/t) = r_I(t) \int G_I d\nu_I + o\{r_I(t)\}. \quad (5.5)$$

If the hypotheses hold on every retained face and every omitted face satisfies

$$\mathbb{E}|G_I(\mathbf{X}_I/t)| = o\{r(t)\}, \quad (5.6)$$

then [Lemma 5.1](#) gives the expansion at target rate $r(t)$ by adding the retained face integrals. One sufficient route to [Eq. \(5.6\)](#) is the same transfer and uniform-integrability hypotheses on that face together with $r_I(t) = o\{r(t)\}$; the nominal vague scale alone is not sufficient.

Product-power hidden measures, including Gaussian ones, permit an explicit integrability test.

Proposition 5.3 (Product-power pair functional). *Suppose $I = \{i, j\}$, $0 < a_i, a_j < 1$, and*

$$v_I(d\mathbf{x}_I) = K_I a_i a_j x_i^{-a_i-1} x_j^{-a_j-1} dx_i dx_j. \quad (5.7)$$

Then $G_I \in L^1(v_I)$ and, with continuous interpretation when $a_i + a_j = 1$,

$$\int G_I dv_I = -K_I w_i^{a_i} w_j^{a_j} \frac{\Gamma(1-a_i)\Gamma(1-a_j)}{\Gamma(1-a_i-a_j)}. \quad (5.8)$$

In particular, a noncritical hidden face gives a pure power correction. For faces of dimension at least three, individual conditions $a_i < 1$ do not suffice: partial sums on proper subfaces control integrability. Equality at one can produce a logarithm; a larger sum can produce polynomial divergence.

Remark (Critical-face aggregation). This phrase means that the first non-negligible face correction lies at the boundary of integrability. It is the geometric reason for logarithms such as the independent $(\log t)/t^2$ term. If the hidden density is integrable against G_I , one obtains a power correction. At the critical boundary a truncated integral grows logarithmically; beyond it, the proper subfaces must be subtracted and analyzed first. Closely related bivariate integrable/nonintegrable edge classifications and the independent critical logarithm are already developed for heavy-tailed copula sums by Yang and Zhang [\[42\]](#). The purpose here is to embed that phenomenon in the full coordinate-face lattice and translate it to harmonic-mean and positive Half-Cauchy calibration.

5.1 Critical local-corner calibration near independence

The general face calculus in the preceding section becomes operational when the pairwise copula densities have finite limits at the lower corner. This subsection isolates that near-independence regime and turns the local corner limits into explicit logarithmic tail and calibration coefficients. The independent logarithm is one instance of the broader critical-pair phenomenon. In dimension at least three, however, a theorem must also control higher Möbius faces: pairwise margins alone cannot exclude a singular multivariate component.

Lemma 5.4 (Product-corner envelope for higher face increments). *For $k \geq 2$, put*

$$\Gamma_k(\mathbf{z}) = \sum_{J \subseteq [k]} (-1)^{k-|J|} \mathbf{1}\left\{\sum_{j \in J} z_j > 1\right\}, \quad \mathbf{z} \in (0, \infty)^k.$$

For $0 < \varepsilon < 1/k$, there is a constant $K_k < \infty$ such that

$$\int_{[\varepsilon, \infty)^k} |\Gamma_k(\mathbf{z})| \prod_{j=1}^k z_j^{-2} d\mathbf{z} \leq \begin{cases} K_2 \{1 + \log(1/\varepsilon)\}, & k = 2, \\ K_k \varepsilon^{-(k-2)}, & k \geq 3. \end{cases} \quad (5.9)$$

Consequently, a bounded product-type density can produce logarithmic pair-face divergence, whereas each fixed face of dimension at least three contributes at most $O(t^{-2})$ after nominal t^{-k} scaling.

Theorem 5.5 (Sharp critical-pair calibration). *Let $\mathbf{U} = (U_1, \dots, U_m)$ have exact uniform margins, let $\mathbf{w}$ be a fixed simplex weight vector with active weights strictly positive, and put $Y_i = U_i^{-1}$.*

For every $I \subseteq [m]$ with $|I| \geq 2$, suppose the marginal copula of $\mathbf{U}_I$ has a density c_I . Assume that, for some $u_0 > 0$ and constants $M_I < \infty$,

$$\operatorname{ess\,sup}_{\substack{\mathbf{u}_I \in (0,1)^I \\ \min_{i \in I} u_i \leq u_0}} c_I(\mathbf{u}_I) \leq M_I. \quad (5.10)$$

For each pair $i < j$, assume that the finite lower-corner limit

$$\lambda_{ij} = \lim_{s \downarrow 0} \operatorname{ess\,sup}_{0 < u, v \leq s} c_{ij}(u, v) = \lim_{s \downarrow 0} \operatorname{ess\,inf}_{0 < u, v \leq s} c_{ij}(u, v) \quad (5.11)$$

exists. Define

$$L_{\mathbf{w}}^{(0)} = \sum_{i < j} \lambda_{ij} w_i w_j. \quad (5.12)$$

Then

$$\mathbb{P}\left(\sum_{i=1}^m w_i Y_i > t\right) = \frac{1}{t} + 2L_{\mathbf{w}}^{(0)} \frac{\log t}{t^2} + o\left(\frac{\log t}{t^2}\right). \quad (5.13)$$

More precisely, let

$$\omega_{ij}(s) = \operatorname{ess\,sup}_{0 < u, v \leq s} |c_{ij}(u, v) - \lambda_{ij}|.$$

If every corner modulus is Dini integrable,

$$\int_0^{u_0} \frac{\omega_{ij}(s)}{s} ds < \infty, \quad i < j, \quad (5.14)$$

then

$$\mathbb{P}\left(\sum_{i=1}^m w_i Y_i > t\right) = \frac{1}{t} + 2L_{\mathbf{w}}^{(0)} \frac{\log t}{t^2} + O(t^{-2}). \quad (5.15)$$

The Dini condition holds, in particular, if $c_{ij}(u, v) - \lambda_{ij} = O\{(u + v)^\eta\}$ for some $\eta > 0$.

Let

$$H_i = \cot\left(\frac{\pi U_i}{2}\right), \quad c = \frac{2}{\pi}.$$

Under the same assumptions,

$$\mathbb{P}\left(\sum_{i=1}^m w_i H_i > t\right) = \frac{c}{t} + 2c^2 L_{\mathbf{w}}^{(0)} \frac{\log t}{t^2} + o\left(\frac{\log t}{t^2}\right), \quad (5.16)$$

and the final remainder is $O(t^{-2})$ under [Eq. \(5.14\)](#).

Remark (What is really required of higher faces). The density condition [Eq. \(5.10\)](#) for $|I| \geq 3$ is a convenient sufficient condition, not a necessary one. The proof only requires

$$\sum_{|I| \geq 3} |\mathbb{E} G_I(\mathbf{Y}_I/t)| = O(t^{-2}).$$

Thus it may be replaced by model-specific control of higher Möbius increments. The condition is vacuous in the bivariate case, but pairwise density bounds do not control higher-dimensional singular components.

Corollary 5.6 (Critical-corner calibration relative to independence). *For $S \in \{R, T\}$, put $c_R = 1$, $c_T = c = 2/\pi$. Let $\bar{F}_{C,w}^S$ and $\bar{F}_{\Pi,w}^S$ denote the tails of S_w under copula C and under independence, respectively, and let $q_{C,w}^S$ and $q_{\Pi,w}^S$ be their upper quantiles. Define $\text{Cal}_{C,w}^S$ using the independence quantile as in Eq. (4.3). Under the hypotheses of Theorem 5.5, for both the reciprocal/harmonic-mean and positive Half-Cauchy aggregates,*

$$\bar{F}_{C,w}^S(t) - \bar{F}_{\Pi,w}^S(t) = 2c_S^2 \sum_{i < j} (\lambda_{ij} - 1) w_i w_j \frac{\log t}{t^2} + o\left(\frac{\log t}{t^2}\right), \quad (5.17)$$

$$\text{Cal}_{C,w}^S(\alpha) = 2 \sum_{i < j} (\lambda_{ij} - 1) w_i w_j \alpha^2 \log(1/\alpha) + o\{\alpha^2 \log(1/\alpha)\}. \quad (5.18)$$

Under the Dini condition, the two final remainders are respectively $O(t^{-2})$ and $O(\alpha^2)$, and

$$q_{C,w}^S(\alpha) - q_{\Pi,w}^S(\alpha) = 2c_S \sum_{i < j} (\lambda_{ij} - 1) w_i w_j \log(1/\alpha) + O(1). \quad (5.19)$$

For common bivariate copulas with an interior parameter, the theorem gives the following lower-corner multipliers: The corresponding densities are analytic or polynomial at the lower corner for fixed interior

Table 1: Lower-corner density multipliers for common bivariate copulas.

Copula	$\lambda = c_C(0, 0)$	Copula	$\lambda = c_C(0, 0)$
product	1	Frank with $\theta \neq 0$	$\theta/(1 - e^{-\theta})$
FGM with parameter θ	$1 + \theta$	Plackett with $\theta > 0$	θ
Ali-Mikhail-Haq with $\theta < 1$	$(1 - \theta)^{-1}$		

parameters, so the Dini condition holds. Finite Bernstein and checkerboard copulas are also covered whenever their lower-corner cell density has a well-defined limit. These specializations should be viewed against the existing second-order risk-aggregation literature [12, 24]; the theorem’s intended contribution is the face-calculus formulation and the harmonic-mean/positive-Half-Cauchy calibration coefficient, not a priority claim for every bivariate special case.

6 The second-order landscape inside index-one MRV

It is useful to standardize terminology before treating the models. If S_w has leading tail constant c_0 , write

$$t\mathbb{P}(S_w > t) = c_0 + E_w(t), \quad E_w(t) \sim K_w t^{-\kappa} L(t). \quad (6.1)$$

Here L is *slowly varying*:

$$\frac{L(tx)}{L(t)} \longrightarrow 1 \quad \text{for every fixed } x > 0. \quad (6.2)$$

Constants, powers of $\log t$, and powers of $\log \log t$ are typical examples. The absolute second tail term in Eq. (6.1) is $K_w t^{-1-\kappa} L(t)$. A smaller positive κ means slower convergence. At the same κ , a growing L is slower and a decaying L is faster than a pure power.

Support and asymptotic order answer different questions. The support of μ records where first-order extremes occur. We use the axial, interior, proper-face, and mixed categories defined by the charged strata in Eq. (2.9) and illustrated in Fig. 1. The word “face-supported” is sometimes used broadly enough to include axes; here it always means a nontrivial proper coordinate face unless stated otherwise.

A support category does not determine a rate. Within each category below we therefore distinguish perturbations of strata already charged by μ , hidden mass on uncharged strata, the intrinsic score-map scale, and rate-free convergence. Every hidden contribution is integrated against its Möbius half-space increment and classified as finite, critically divergent, or polynomially divergent. The slowest valid absolute rate wins, tied terms add, and score-map terms are compared afterward. This is an audit framework rather than a completed recursive theorem for all nested faces.

6.1 Axis-supported copulas

Axis support means that ordinary MRV sees only one-coordinate extremes. It says merely that every comparable multi-coordinate extreme is $o(t^{-1})$; it does not determine its smaller rate. In the present exact p -value setting, the first possible ordinary axis mechanism can be largely removed before studying the missing faces.

What exact marginal normalization removes The reciprocal and Half-Cauchy scores must be distinguished at second order. Exact reciprocal scores have no marginal tail correction at all, whereas the Half-Cauchy map has an intrinsic third-order tail term.

Exact Pareto standardization forces every marginal projection of an ordinary signed second-order measure to vanish; if that measure is supported only on the axes, it is zero. In contrast, exact Half-Cauchy margins have

$$t\mathbb{P}(H_i > tx) = \frac{c}{x} - \frac{c}{3x^3}t^{-2} + O(t^{-4}), \quad c = \frac{2}{\pi},$$

so the score transformation itself contributes at absolute order t^{-3} . The formal signed-measure statement is in [Appendix A](#).

Thus a nonzero second term for the exact reciprocal sum cannot be a purely marginal radial correction on the axes. It must involve dependence, angular redistribution, a hidden or boundary face, or a failure of a regular expansion. This does *not* mean that every quantitative main-cone correction vanishes: a signed measure can have zero marginal projections and still be nonzero on a weighted half-space. For T_w the t^{-3} score-map correction is always a candidate, whereas R_w has no marginal score-map correction; both are often dominated, for example, by the independent $(\log t)/t^2$ interaction term.

The face integral and the integrable–critical–nonintegrable trichotomy For two positive weights, the pair increment is

$$G_{12}(x_1, x_2) = \mathbf{1}\{w_1x_1 + w_2x_2 > 1\} - \mathbf{1}\{w_1x_1 > 1\} - \mathbf{1}\{w_2x_2 > 1\}. \quad (6.3)$$

The exact identity

$$\begin{aligned} \mathbb{P}(w_1X_1 + w_2X_2 > t) &= \mathbb{P}(w_1X_1 > t) + \mathbb{P}(w_2X_2 > t) \\ &\quad + \mathbb{E}G_{12}(X_1/t, X_2/t) \end{aligned} \quad (6.4)$$

shows that G_{12} counts cooperative exceedances positively and removes double-counted simultaneous marginal exceedances negatively. If a hidden measure ν_{12} appears at absolute scale $r_{12}(t)$, the number

$$\int G_{12} d\nu_{12} \quad (6.5)$$

is the *face integral*. When it is absolutely integrable and the uniform-integrability condition of [Theorem 5.2](#) holds, it is the coefficient that transfers the hidden probability scale to the sum tail. If that absolute scale is

$r_I(t) = t^{-q_I} L_I(t)$, then its relative calibration exponent is $q_I - 1$. Integrability transfers the scale as it stands; it does not itself create a logarithm.

The difficulty is that the open face $(0, \infty]^2$ is not separated from its axes. Values such as $x_2 \downarrow 0$ represent an original score $X_2 = o(t)$ that may nevertheless range through all intermediate scales.

For a symmetric product-power face density with exponent a , truncation at distance ε from the axes gives

$$\int |G_{12}| dv_a = \begin{cases} O(1), & 0 < a < 1, \\ \Theta\{\log(1/\varepsilon)\}, & a = 1, \\ \Theta(\varepsilon^{1-a}), & a > 1. \end{cases}$$

Thus the three regimes are integrable, critical, and polynomially nonintegrable. A formal proposition is given in [Appendix A](#).

This proposition places independence and Gaussian copulas on one continuous boundary diagram. For independent unit-Pareto margins, the hidden pair density is $x_1^{-2}x_2^{-2}$, so $a = 1$, the nominal hidden scale is t^{-2} , and the effective cutoff $\varepsilon \asymp t^{-1}$ produces $t^{-2} \log t$. For a one-sided Gaussian copula with $0 < \rho < 1$,

$$a_\rho = \frac{1}{1+\rho} < 1, \quad q_\rho = 2a_\rho = \frac{2}{1+\rho}, \quad \beta_\rho = 1 - a_\rho = \frac{\rho}{1+\rho}. \quad (6.6)$$

Its hidden probability scale is

$$r_\rho(t) = t^{-q_\rho} (\log t)^{-\beta_\rho}. \quad (6.7)$$

Here q_ρ is the power exponent, β_ρ is the exponent of the slowly varying Gaussian-quantile factor, and a_ρ controls spatial integrability near a proper face. The logarithmic factor in [Eq. \(6.7\)](#) is already part of the Gaussian joint-tail scale; it is not created by the face integral. Since $a_\rho < 1$, the face integral is finite; when the boundary-transfer condition holds, the sum correction has the same order as $r_\rho(t)$. Its relative exponent is

$$\kappa_G = q_\rho - 1 = \frac{1-\rho}{1+\rho}. \quad (6.8)$$

At $\rho = 0$, $a_\rho = 1$ and the face integral becomes critical, producing the growing independent logarithm. Here “critical” means failure at the boundary of integrability, not “logarithmic by definition”; nested boundaries can generate other slowly varying amplifications. For $-1 < \rho < 0$, $a_\rho > 1$ and the formal lower-lower hidden face is nonintegrable. This is a concrete nonintegrable-face example, but not a completed aggregation theorem: proper edge regions must first be resolved, and the one-sided nonpositive Gaussian expansion with at least one strictly negative correlation remains open. The all-zero boundary is independence and is already proved. The standard- t angular boundary at $\nu = 1$ is another critical example, while $0 < \nu < 1$ is nonintegrable; those cases also remain open.

Singular exclusion geometry Hidden positive mass is not the only possible axis-supported mechanism. For the countermonotone copula $(U_1, U_2) = (U, 1 - U)$, comparable lower extremes are exactly excluded. With weights $(w, 1 - w)$, direct solution of the quadratic boundary gives

$$\mathbb{P}(R_w > t) = 1 - \frac{\sqrt{t^2 - 2t + (1 - 2w)^2}}{t} = \frac{1}{t} + \frac{2w(1 - w)}{t^2} + O(t^{-3}). \quad (6.9)$$

For the positive Half-Cauchy scores, writing $H = \cot(\pi U/2)$ gives the other score as H^{-1} and

$$T_w = wH + (1 - w)H^{-1}.$$

The exact calculation in Eq. (B.31) yields

$$\mathbb{P}(T_w > t) = \frac{2}{\pi t} + \frac{2}{\pi} \left\{ 2w(1-w) - \frac{1}{3} \right\} t^{-3} + O(t^{-5}). \quad (6.10)$$

Thus the independent critical logarithm disappears for both aggregates. The reciprocal statistic retains an order- t^{-2} dependence correction, whereas for the positive Half-Cauchy statistic that coefficient cancels and the t^{-3} transformation scale becomes visible. This is a singular-support cancellation/exclusion mechanism, not an integrable hidden-pair correction.

Rate-free and other possibilities Ordinary MRV alone supplies no numerical rate. Appendix A gives an exact-uniform copula with axial lower MDA whose measure-level convergence is slower than every positive power and has no nonzero regularly varying diagonal normalization. Rate-free convergence can occur under any support category, although it is especially important here because axial first-order support otherwise tempts one to infer a hidden power rate. Signed cancellations, several hidden faces with tied rates, or a copula singularity near an opposite corner can likewise determine the first omitted term. Axis support alone does not choose among these possibilities.

6.2 Interior-supported copulas

Interior support means that the ordinary exponent measure already sees simultaneous extremes. Consequently the standard hidden-regular-variation question “at what smaller scale do several coordinates become large together?” does not arise.

Proposition 6.1 (No hidden subset joint-extreme cone under interior support). *Suppose S is a nonzero spectral measure concentrated on $F_{[m]}^\circ$. For every nonempty $I \subseteq [m]$,*

$$\mu\{\mathbf{x} : x_i > 1 \text{ for every } i \in I\} = \int_{F_{[m]}^\circ} \min_{i \in I} s_i S(ds) > 0. \quad (6.11)$$

Hence every fixed subset joint exceedance has first-order probability:

$$\mathbb{P}(X_i > t \text{ for every } i \in I) \sim \frac{1}{t} \int \min_{i \in I} s_i S(ds). \quad (6.12)$$

A smaller-scale limit concentrated near an exact boundary face may still describe an angular boundary layer, but it is not automatically an additive second term. Neighborhoods of that face have already been integrated as part of the ordinary interior measure. One must first subtract the full first-order contribution and prove a quantitative boundary expansion. Thus the core second-order problem for an interior-supported copula is ordinarily a perturbation of mass already present—radial, angular, or both—rather than the first appearance of a joint-extreme face. Boundary integrability can still fail after this subtraction, as in the standard- t cases $\nu \leq 1$.

This is the canonical setting for ordinary second-order MRV: a signed perturbation of the main-cone measure is evaluated on the weighted half-space through the quantitative projection in Appendix A. Exact Pareto margins force its one-coordinate projections to vanish, but they do not force its joint half-space coefficient to vanish.

The common-factor models in this paper make the dominant mechanism explicit. The following elementary power transformation explains why their score-scale radial variable has index one.

Lemma 6.2 (Homogeneous power converts the common radial index to one). *Suppose*

$$\mathbb{P}(R > r) = ar^{-\alpha}\{1 - br^{-\beta} + o(r^{-\beta})\}, \quad \alpha, \beta > 0. \quad (6.13)$$

Put $Q = R^\alpha$. Then

$$\mathbb{P}(Q > q) = aq^{-1}\{1 - bq^{-\beta/\alpha} + o(q^{-\beta/\alpha})\}. \quad (6.14)$$

If the leading score map is homogeneous of degree α in R , it can therefore be written at leading order as $QA(\Theta)$ with an index-one common radial factor.

The support-agnostic common-factor inversion theorem in [Appendix A](#) converts a second-order radial tail, an angular score expansion, and the intrinsic score-map expansion into an aggregate-tail coefficient. Its central rule is that the smallest relative exponent wins and tied radial, angular, and score-map terms add. The theorem also states the dominated-inversion conditions needed near angular subfaces.

For the standard multivariate- t_ν model, the transformed radial factor is $Q = R_0^\nu$ and its relative second-order exponent is $2/\nu$. Two-sided scores have a strictly positive angular vector and hence interior support; one-sided scores are mixed-supported because their positive-part angular vector can lie on lower-dimensional faces. The same inversion theorem applies to both orientations. For $\nu > 1$ it yields the ordinary radial correction at absolute order $t^{-1-2/\nu}$, with the additional one-sided reciprocal angular term described in [Theorem 8.1](#).

For positive Clayton, $Q = V^{-1/\theta}$ already has index one, its relative second-order exponent is θ , and the exponential angular coordinates are strictly positive. The model is therefore interior-supported. The reciprocal correction has absolute order $t^{-1-\theta}$, while the positive Half-Cauchy correction meets its score-map scale at $\theta = 2$ and is governed by that scale for $\theta > 2$; see [Theorems 8.2](#) and [8.3](#). The formal standard- t collision occurs at $\nu = 1$, where angular integrability also becomes critical.

6.3 Proper-face-supported copulas

Genuine proper faces first appear when $m \geq 3$. Marshall–Olkin and max-linear common-shock copulas can place spectral atoms on selected proper-face rays, while Tawn asymmetric logistic models can place diffuse mass on selected faces [\[17, 18\]](#). The required lower-tail orientation must be stated explicitly. A concrete three-dimensional max-linear construction and its exact-uniform transformation are recorded in [Appendix A](#).

A sharp proper-face pair-shock theorem The max-linear construction in [Appendix A](#) admits an exact shock representation on the p -value scale. This is a spectrally discrete max-linear model in the sense discussed, for example, by Wang and Stoev [\[40\]](#). Let E_{12}, E_{13}, E_{23} be independent $\text{Exp}(1)$ variables and put

$$T_1 = \min(E_{12}, E_{13}), \quad T_2 = \min(E_{12}, E_{23}), \quad T_3 = \min(E_{13}, E_{23}),$$

together with

$$U_i = 1 - \exp(-2T_i), \quad i = 1, 2, 3. \quad (6.15)$$

Since each T_i is $\text{Exp}(2)$, every U_i is exactly uniform. Moreover,

$$\mathbb{P}(U_1 > u_1, U_2 > u_2, U_3 > u_3) = \prod_{i < j} \{1 - \max(u_i, u_j)\}^{1/2}. \quad (6.16)$$

Thus this is the lower-tail, or reflected, orientation of a Marshall–Olkin pair-shock copula. For a simplex weight vector define

$$\sigma_2(\mathbf{w}) = \sum_{i < j} w_i w_j. \quad (6.17)$$

The auxiliary three-edge critical-integral lemma in [Appendix A](#) shows that two independent edge shocks create a $t^{-2} \log t$ correction with an explicit coefficient.

Theorem 6.3 (Proper-face pair-shock aggregation). *Let U be given by Eq. (6.15) and suppose $w_i > 0$ with $\sum_i w_i = 1$. Set*

$$Y_i = U_i^{-1}, \quad H_i = \cot(\pi U_i/2), \quad c = \frac{2}{\pi}.$$

Then the spectral measure of Y is

$$S = \delta_{(1/2, 1/2, 0)} + \delta_{(1/2, 0, 1/2)} + \delta_{(0, 1/2, 1/2)}. \quad (6.18)$$

In particular, it is supported purely on nontrivial proper coordinate faces. Furthermore,

$$\mathbb{P}\left(\sum_{i=1}^3 w_i Y_i > t\right) = \frac{1}{t} + \sigma_2(\mathbf{w}) \frac{\log t}{t^2} + O(t^{-2}), \quad (6.19)$$

$$\mathbb{P}\left(\sum_{i=1}^3 w_i H_i > t\right) = \frac{c}{t} + c^2 \sigma_2(\mathbf{w}) \frac{\log t}{t^2} + O(t^{-2}). \quad (6.20)$$

Corollary 6.4 (Proper-face calibration errors). *For $S \in \{R, T\}$, set $c_R = 1$, $c_T = c$, let $F_{S, \mathbf{w}}^\Pi$ be the corresponding independent-reference distribution with the same weights, and let $q_{\Pi, S, \mathbf{w}}(\alpha)$ be its upper α -quantile. Under the pair-shock model,*

$$\mathbb{P}(S_{\mathbf{w}} > t) - \{1 - F_{S, \mathbf{w}}^\Pi(t)\} = -c_S^2 \sigma_2(\mathbf{w}) \frac{\log t}{t^2} + O(t^{-2}). \quad (6.21)$$

Consequently, as $\alpha \downarrow 0$,

$$\mathbb{P}\{S_{\mathbf{w}} > q_{\Pi, S, \mathbf{w}}(\alpha)\} = \alpha - \sigma_2(\mathbf{w}) \alpha^2 \log(1/\alpha) + O(\alpha^2), \quad (6.22)$$

whereas at the simpler first-order threshold c_S/α ,

$$\mathbb{P}(S_{\mathbf{w}} > c_S/\alpha) = \alpha + \sigma_2(\mathbf{w}) \alpha^2 \log(1/\alpha) + O(\alpha^2). \quad (6.23)$$

The interpretation is recursive. One large shock supplies the proper-face first-order exponent measure. Two independent edge shocks supply hidden full-interior mass of order t^{-2} , and its accumulation near the already-charged edge rays is critical, producing the logarithm. The coefficient is exactly half the independent-coordinate coefficient. This is a sharp theorem for the symmetric three-dimensional pair-shock model, not a general second-order theorem for all proper-face-supported copulas.

The second-order problem in this category can combine three directions of movement: quantitative perturbation on the already-charged proper faces, hidden mass on larger faces not charged at first order, and nonuniform approach to smaller subfaces. The weighted half-space reaches all of them. This is the natural setting for recursive Möbius subtraction. Nested critical boundaries can create higher powers of logarithms or other slowly varying amplifications; polynomially divergent face integrals require the adjacent smaller faces to be subtracted and analyzed first. No general theorem yet covers all such nested rates.

6.4 Mixed-support copulas

Mixed support is common once $m \geq 3$; representative examples are:

1. *One-sided standard multivariate- t scores.* Their leading angular vector is proportional to

$$((G_1^+)^v, \dots, (G_m^+)^v). \quad (6.24)$$

For a positive-definite correlation matrix every Gaussian sign orthant has positive probability. Hence different sign patterns generate axes, every available proper coordinate face, and the full interior. This is a mixed-support model, whereas the two-sided version is interior-supported.

2. *Extremal- t copulas* [19]. Their positive-part Gaussian spectral representation yields the same sign-pattern mechanism and hence a natural mixture of coordinate strata.
3. *Marshall–Olkin/max-linear and Tawn asymmetric logistic models*. Including shocks or logistic components indexed by subsets of different sizes produces a mixture of axis atoms, proper-face components, and possibly full-interior mass.
4. *Convex mixtures of copulas in a common lower MDA*. If exact-uniform copulas C_k are mixed with fixed probabilities π_k , their exponent measures mix as

$$\mu = \sum_k \pi_k \mu_k. \quad (6.25)$$

Mixing, for example, an axis-supported Gaussian copula with an interior-supported Clayton copula gives a direct mixed-support model.

Mixed support need not always be analyzed one face at a time. For $\nu > 1$, the common-radial representation gives the one-sided standard- t expansions for both $R_{\mathbf{w}}$ and $T_{\mathbf{w}}$ by integrating over the whole Gaussian angular law; their second terms differ when the bounded one-sided angular contribution becomes visible. Face recursion becomes indispensable when charged strata have different quantitative rates, absent strata appear at hidden scales, or angular integrals fail near proper subfaces. A general second-order mixed-support theorem is not claimed here.

Why axes and full interior remain the canonical poles Axis support and full-interior support are not exhaustive in dimension at least three, but they are defensible organizing poles for this problem. Axis-supported models represent asymptotic independence and include independence, Gaussian copulas, and many smooth lower-tail-independent families. Their central second-order question is how dependence first reappears on hidden faces. Full-interior models represent asymptotic dependence and include positive Clayton and two-sided standard- t scores. Their central question is how already-visible joint mass approaches its radial-angular limit.

Proper-face and mixed-support models describe clustered or blockwise extremes and cannot be dismissed as pathological. They form the natural intermediate theory. The four subsections above are therefore an organizing principle, not a claim that the axial and full-interior poles exhaust multivariate copulas.

The four support categories classify where the first-order exponent measure lives. The mechanisms in Table 2 instead classify how the next contribution reaches the weighted half-space. These are crossed classifications: first-order support alone does not determine the second-order mechanism. Singular exclusion, a pure score-map correction, and rate-free convergence are additional possibilities beyond the four principal measure-transfer mechanisms in the table.

Table 2: Four principal second-order measure-transfer mechanisms.

Mechanism	Explanation	Examples
Ordinary second-order MRV	On the main cone, $t\mathbb{P}(X/t \in \cdot) = \mu(\cdot) + A(t)\nu_2(\cdot) + o\{A(t)\}$. When the signed perturbation is integrable on the weighted half-space, it produces an absolute tail correction of order $t^{-1}A(t)$. This mechanism perturbs mass already visible at first order.	Positive Clayton and standard multivariate- t_ν , $\nu > 1$, through the proved common-factor expansions; one-sided reciprocal t aggregation also has an angular competition.

Mechanism	Explanation	Examples
Integrable hidden face	The first-order measure gives no mass to a face I , but a hidden measure appears at an absolute scale $r_I(t) = o(t^{-1})$. If $\int G_I dv_I < \infty$, together with the required uniform integrability, the aggregate inherits the same scale $r_I(t)$; the face integration creates no additional logarithm.	One-sided Gaussian copulas with an active positive correlation and two-sided Gaussian copulas with a nonzero active correlation. Their compact hidden-face limits are known; the weighted-half-space coefficient is conditional on the explicit boundary hypotheses in Appendix A .
Critical hidden face	The hidden-face integral fails exactly at its integrability boundary. A finite-threshold cut-off against adjacent subfaces then produces a slowly varying amplification. One simple critical boundary gives $\log t$, although nested boundaries can yield other slowly varying factors.	Product copula and the three-edge pair-shock model; their $t^{-2} \log t$ terms are proved.
Nonintegrable hidden face	The truncated face integral diverges polynomially. The nominal hidden rate cannot be transferred directly to the sum tail: adjacent lower-dimensional strata must first be subtracted and analyzed through the recursive Möbius increments.	The formal lower-lower hidden face of a one-sided Gaussian copula with negative correlation; the diagnosis is established, but its leading aggregation expansion remains open.

[Table 2](#) is the conceptual map. The full model-by-model ledger in [Table 5](#) in [Appendix B](#) records first-order support, absolute second-order rate, mechanism, proof status, and bounded-density diagnostics.

7 Gaussian copulas

7.1 One-sided p -values

Let $\mathbf{Z} = (Z_1, \dots, Z_m)$ be centered Gaussian with unit variances and correlation matrix $R = (\rho_{ij})$. Assume

$$\rho_{\max} = \max_{i \neq j} |\rho_{ij}| < 1. \quad (7.1)$$

Define the one-sided p -values

$$U_i = 1 - \Phi(Z_i).$$

Theorem 7.1 (Gaussian-copula score vector). *Under [Eq. \(7.1\)](#), the reciprocal scores $Y_i = \{1 - \Phi(Z_i)\}^{-1}$ and positive Half-Cauchy scores $H_i = \cot[\pi\{1 - \Phi(Z_i)\}/2]$ are index-one MRV with axis-supported tail measure. Hence [Theorem 2.3](#) applies to both R_w and T_w .*

This proof avoids applying an equal-threshold bivariate-normal asymptotic to unequal thresholds. It also makes clear why strict inequality in [Eq. \(7.1\)](#) is required: correlation one can produce first-order simultaneous extremes, although that case may still be MRV with a non-axis-supported measure.

7.2 Two-sided p -values

For

$$U_i = 2\{1 - \Phi(|Z_i|)\},$$

the pair event is the union of four sign-quadrant events. For signs $\varepsilon_i, \varepsilon_j \in \{-1, 1\}$, the pair $(\varepsilon_i Z_i, \varepsilon_j Z_j)$ is Gaussian with correlation $\varepsilon_i \varepsilon_j \rho_{ij}$, whose absolute value is at most $\rho_{\max}$. Applying the pair bound from [Theorem 7.1](#) to all four quadrants and summing proves [Eq. \(3.9\)](#). Thus the two-sided Gaussian reciprocal and positive Half-Cauchy score vectors are also index-one MRV with axis-supported tail measure.

Remark (What is regularly varying). The Gaussian vector $\mathbf{Z}$ itself is not regularly varying; its tails are too light. The transformed vector $(1/U_1, \dots, 1/U_m)$, and hence the Half-Cauchy score vector, is regularly varying. Here MRV concerns the transformed p -values, not the Gaussian observations.

Hidden-pair scale and the independence boundary For a one-sided Gaussian pair with $0 < \rho < 1$, simultaneous reciprocal scores have scale

$$t^{-q_\rho} (\log t)^{-\beta_\rho}, \quad q_\rho = \frac{2}{1+\rho}, \quad \beta_\rho = \frac{\rho}{1+\rho}.$$

The hidden pair density has spatial exponent $a_\rho = (1+\rho)^{-1} < 1$, so its pair-face integral is finite for fixed $\rho > 0$. At $\rho = 0$, $a_\rho = 1$ and the face becomes critical, producing the independent $(\log t)/t^2$ term. The nonuniform crossover occurs at $\rho \log t = O(1)$. The detailed bivariate calculation and the negative-correlation diagnosis are in [Appendix A](#).

Conditional fixed-dimensional expansion Let ρ_* be the largest active positive correlation for one-sided p -values, or the largest active absolute correlation for two-sided p -values. Under the explicit face-uniform-integrability and higher-face remainder conditions stated in [Appendix A](#), the reciprocal and positive Half-Cauchy tails have the respective forms

$$\begin{aligned} \mathbb{P}(R_{\mathbf{w}} > t) &= t^{-1} + D_{G,\mathbf{w}} t^{-q_{\rho_*}} (\log t)^{-\beta_{\rho_*}} + o\{t^{-q_{\rho_*}} (\log t)^{-\beta_{\rho_*}}\}, \\ \mathbb{P}(T_{\mathbf{w}} > t) &= ct^{-1} + c^{q_{\rho_*}} D_{G,\mathbf{w}} t^{-q_{\rho_*}} (\log t)^{-\beta_{\rho_*}} + o\{t^{-q_{\rho_*}} (\log t)^{-\beta_{\rho_*}}\}, \quad c = 2/\pi. \end{aligned}$$

The coefficient is an explicit sum over the maximally correlated pairs. The bivariate hidden vague limit is proved, but the required truncation at adjacent axes and the higher-face remainder are not yet verified in full dimension. This is therefore a conditional weighted-half-space result, not an unconditional Gaussian theorem; it is also nonuniform at both the independence and perfect-correlation boundaries. The one-sided problem with nonpositive correlations remains open.

8 Common-factor copulas: multivariate- t and positive Clayton

This section specializes the common-factor machinery of [Section 6.2](#) first to the standard multivariate- t copula and then to positive Clayton copulas.

For the standard common-scale multivariate- t_ν copula, both one- and two-sided reciprocal and positive Half-Cauchy score vectors are index-one MRV. Two-sided scores have interior spectral mass; the one-sided positive-part angular vector produces mixed support. The first-order proof, the common-scale representation, and the usual tail-dependence formula are collected in [Appendix A](#).

Radial source of the second-order rate The transformed radial factor has index one and relative second-order exponent $2/\nu$, yielding candidate aggregate order $t^{-1-2/\nu}$. For one-sided reciprocal scores it competes with a bounded angular term at $\nu = 2$; see [Appendix A](#).

8.1 Second-order theorem for the standard multivariate- t copula

The preceding qualification is resolved here for the standard common-scale model. For $\mathbf{G} \sim N_m(0, R)$, define

$$a_\nu = \frac{\nu^{\nu/2}}{2^{\nu/2}\Gamma(\nu/2 + 1)}, \quad b_\nu = \frac{\nu^2}{2(\nu + 2)}, \quad (8.1)$$

$$k_\nu = \frac{\Gamma((\nu + 1)/2)\nu^{\nu/2-1}}{\sqrt{\pi}\Gamma(\nu/2)}, \quad d_\nu = \frac{\nu^2(\nu + 1)}{2(\nu + 2)}. \quad (8.2)$$

$$A_+ = k_\nu^{-1} \sum_i w_i (G_i^+)^{\nu}, \quad C_- = \sum_i w_i \mathbf{1}\{G_i < 0\}, \quad (8.3)$$

$$B_+ = d_\nu k_\nu^{-1} \sum_i w_i \mathbf{1}\{G_i > 0\} G_i^{\nu-2}, \quad (8.4)$$

$$A_{|\cdot|} = (2k_\nu)^{-1} \sum_i w_i |G_i|^\nu, \quad B_{|\cdot|} = d_\nu (2k_\nu)^{-1} \sum_i w_i |G_i|^{\nu-2}. \quad (8.5)$$

$$D_{\nu, \mathbf{w}}^s = a_\nu \mathbb{E}[(A_s)^{2/\nu} (B_s - b_\nu A_s)], \quad s \in \{+, |\cdot|, \cdot\}. \quad (8.6)$$

Theorem 8.1 (Second-order standard multivariate- t aggregation). *Fix m , a correlation matrix R , simplex weights $\mathbf{w}$, and $\nu > 1$. Let $\mathbf{Z}$ have the standard multivariate- t_ν distribution with correlation matrix R , and let T_ν denote the univariate standard- t_ν CDF.*

For one-sided reciprocal scores $Y_i = \{1 - T_\nu(Z_i)\}^{-1}$,

$$\mathbb{P}\left(\sum_i w_i Y_i > t\right) = \begin{cases} t^{-1} + D_{\nu, \mathbf{w}}^+ t^{-1-2/\nu} + o(t^{-1-2/\nu}), & \nu > 2, \\ t^{-1} + \{D_{2, \mathbf{w}}^+ + a_2 \mathbb{E}(A_+ C_-)\} t^{-2} + o(t^{-2}), & \nu = 2, \\ t^{-1} + a_\nu \mathbb{E}(A_+ C_-) t^{-2} + o(t^{-2}), & 1 < \nu < 2. \end{cases} \quad (8.7)$$

For two-sided reciprocal scores $Y_i = [2\{1 - T_\nu(|Z_i|)\}]^{-1}$,

$$\mathbb{P}\left(\sum_i w_i Y_i > t\right) = t^{-1} + D_{\nu, \mathbf{w}}^{|\cdot|} t^{-1-2/\nu} + o(t^{-1-2/\nu}). \quad (8.8)$$

For either one- or two-sided Half-Cauchy scores, with the corresponding choice of s ,

$$\mathbb{P}\left(\sum_i w_i H_i > t\right) = \frac{c}{t} + c^{1+2/\nu} D_{\nu, \mathbf{w}}^s t^{-1-2/\nu} + o(t^{-1-2/\nu}), \quad c = 2/\pi. \quad (8.9)$$

The coefficient is zero for a single active coordinate, as exact marginal uniformity requires.

The displayed standard- t coefficients are dimension-free for fixed $\nu > 1$, but this does not control the fixed-dimensional remainders. The coefficient proposition and the unresolved $\nu \leq 1$ angular boundary are stated in [Appendix A](#).

8.2 Positive Clayton copulas: common-frailty aggregation

For $\theta > 0$, the m -variate Clayton copula is

$$C_{\theta, m}(u_1, \dots, u_m) = \left(\sum_{i=1}^m u_i^{-\theta} - m + 1 \right)^{-1/\theta}. \quad (8.10)$$

It is lower-tail dependent, but it is not a t copula. Its Archimedean frailty representation supplies a particularly transparent common-factor calculation. Let $V \sim \text{Gamma}(1/\theta, 1)$ and, independently, $E_i \stackrel{\text{iid}}{\sim} \text{Exp}(1)$; set $U_i = (1 + E_i/V)^{-1/\theta}$. Then $\mathbf{U}$ has copula [Eq. \(8.10\)](#) and uniform margins. Write $p = 1/\theta$ and $R = V^{-1/\theta}$. The exact reciprocal representation and radial expansion are

$$Y_i = R(E_i + R^{-\theta})^p, \quad (8.11)$$

$$\mathbb{P}(R > r) = a_\theta r^{-1} \{1 - b_\theta r^{-\theta} + O(r^{-2\theta})\}, \quad (8.12)$$

$$a_\theta = \Gamma(1 + 1/\theta)^{-1}, \quad b_\theta = (1 + \theta)^{-1}. \quad (8.13)$$

For simplex weights put $A_{\mathbf{w}} = \sum_i w_i E_i^p$. Write $B_{\mathbf{w}} = p \sum_i w_i E_i^{p-1}$, and let $D_{\theta, \mathbf{w}} = a_\theta \mathbb{E}[A_{\mathbf{w}}^\theta (B_{\mathbf{w}} - b_\theta A_{\mathbf{w}})]$.

Theorem 8.2 (Reciprocal/harmonic-mean aggregation under positive Clayton). *For every fixed m , $\theta > 0$, and simplex weight vector,*

$$\mathbb{P}\left(\sum_i w_i Y_i > t\right) = \frac{1}{t} + D_{\theta, \mathbf{w}} t^{-1-\theta} + o(t^{-1-\theta}). \quad (8.14)$$

Moreover,

$$D_{\theta, \mathbf{w}} = \frac{a_\theta}{\theta + 1} \mathbb{E}\left[(m-1)A_{\mathbf{w}}^{\theta+1} - \sum_{i=1}^m (A_{\mathbf{w}} - w_i E_i^p)^{\theta+1}\right]. \quad (8.15)$$

It is strictly positive when at least two weights are positive and is zero for a single active coordinate.

The Half-Cauchy map has a second intrinsic scale:

$$\cot\left(\frac{\pi}{2Y}\right) = cY - dY^{-1} + O(Y^{-3}), \quad c = \frac{2}{\pi}, \quad d = \frac{\pi}{6}. \quad (8.16)$$

This competes with the Clayton radial correction.

Theorem 8.3 (Positive Half-Cauchy phase transition under Clayton). *Under the conditions of [Theorem 8.2](#), put $Q_{\mathbf{w}} = \sum_i w_i E_i^{-1/\theta}$. Then*

$$\mathbb{P}\left(\sum_i w_i H_i > t\right) = \frac{c}{t} + \begin{cases} c^{1+\theta} D_{\theta, \mathbf{w}} t^{-1-\theta} + o(t^{-1-\theta}), & 0 < \theta < 2, \\ \{c^3 D_{2, \mathbf{w}} - a_2 d c^2 \mathbb{E}(A_{\mathbf{w}}^2 Q_{\mathbf{w}})\} t^{-3} + o(t^{-3}), & \theta = 2, \\ -a_\theta d c^2 \mathbb{E}(A_{\mathbf{w}}^2 Q_{\mathbf{w}}) t^{-3} + o(t^{-3}), & \theta > 2. \end{cases} \quad (8.17)$$

The inverse moment in the last two cases is finite. For one active coordinate, the formula reduces to the exact univariate expansion $\mathbb{P}(H > t) = c/t - c/(3t^3) + O(t^{-5})$.

8.3 Parametric benchmarks

Positive Clayton and standard multivariate- t are both common-factor models after reciprocal-score transformation, but they have different angular laws and second-order exponents: θ for Clayton and $2/\nu$ for standard t . The generic phase diagram, the coefficient bounds, and the explanation of the angularly critical t boundary are given in [Appendix A](#).

The principal parametric benchmarks can now be compared directly. In [Table 3](#), the absolute second-order rate is the order of the first nonzero aggregate-tail term after $1/t$ for $R_{\mathbf{w}}$, or after c/t , $c = 2/\pi$, for $T_{\mathbf{w}}$. Every rate marked proved is fixed-dimensional. For the Gaussian row, $\rho_\star$ denotes the largest active

positive correlation for one-sided p -values and the largest active absolute correlation for two-sided p -values; put $q_\star = \frac{2}{1+\rho_\star}$ and $\beta_\star = \frac{\rho_\star}{1+\rho_\star}$.

Table 3: Principal support, rate, mechanism, and proof-status landscape.

Model	First-order support	Second-order mechanism and current status	Absolute second-order rate
Gaussian, $\rho_\star > 0$	Axes	Integrable hidden pair faces; the candidate coefficient is derived, and the expansion is conditional on the explicit face-transfer conditions in Appendix A .	$t^{-q_\star} (\log t)^{-\beta_\star}$
Product copula	Axes	Critical hidden-pair accumulation; the logarithmic coefficients are proved for R_w and T_w .	$t^{-2} \log t$
Standard multivariate- t , $\nu > 1$	Two-sided: interior; one-sided: mixed	Ordinary common-radial perturbation. For one-sided R_w , a bounded angular term dominates when $1 < \nu < 2$ and ties at $\nu = 2$. Proved in fixed dimension.	T_w and two-sided R_w : $t^{-1-2/\nu}$; one-sided R_w : $t^{-1-2/\nu}$ for $\nu > 2$ and t^{-2} for $1 < \nu < 2$, with a tie at $\nu = 2$
Positive Clayton, $0 < \theta < 2$	Interior	Ordinary common-frailty perturbation. Both fixed- m expansions are proved.	$t^{-1-\theta}$ for R_w and T_w
Positive Clayton, $\theta \geq 2$	Interior	Common frailty for R_w ; for T_w , a tie at $\theta = 2$ and score-map dominance above. Fixed- m result proved.	R_w : $t^{-1-\theta}$; T_w : t^{-3}

No rate is assigned here to one-sided Gaussian models having no positive active correlation and at least one strictly negative correlation, or to standard multivariate- t models with $0 < \nu \leq 1$: their leading second-order aggregation problems remain open. The all-zero Gaussian case is independence and is already settled. The broader family-by-family classification, including conditional mechanism diagnoses and rate-free cases, is given in [Table 5](#) in [Appendix B](#).

9 Discussion

The practical motivation is the absence of exact null calibration for heavily right aggregation under dependence. Harmonic-mean and positive Half-Cauchy scores have universal index-one first-order tails, but $c/t + o(t^{-1})$ does not quantify error at testing levels. Relative to the exact independence law, the next term determines size distortion and the dependence-specific critical-value shift. Its rate, coefficient, and sign therefore govern how quickly first-order validity becomes statistically useful.

The central geometric conclusion is that the weighted half-space reaches every coordinate face. Its correction depends both on where the first-order exponent measure lives and on how finite-threshold mass approaches the face lattice. Ordinary signed perturbations change mass already visible at first order; integrable hidden faces transfer their own scale; critical faces create slowly varying amplification; and nonintegrable faces require recursive subtraction. Support and mechanism are therefore complementary classifications.

The machinery makes this geometry operational. Exact Pareto standardization removes marginal artifacts, quantitative exponent measures identify the scale, Möbius increments decompose the noncompact half-space, and the hidden-face and common-factor theorems turn face or radial-angular expansions into tail coefficients. Calibration inversion then converts those coefficients into rejection-probability and critical-value errors. This is a reusable route from dependence geometry to statistical calibration, rather than a collection of unrelated copula calculations.

The examples show why both layers are needed. Product and Gaussian copulas are axial but lie in critical and integrable hidden-face regimes; positive Clayton and two-sided standard- t perturb interior mass; one-sided standard- t has mixed support; and the pair-shock model gives a sharp proper-face logarithm. [Table 3](#)

compares the principal benchmarks, while the full ledger in [Table 5](#) records support, rate, mechanism, and proof status without turning conditional diagnoses into theorems.

Harmonic-mean and positive Half-Cauchy aggregation share first-order geometry but separate at genuine transformation boundaries. The latter has an intrinsic t^{-3} correction, while one-sided reciprocal t aggregation can have a bounded angular term absent from its Half-Cauchy analogue. Treating the rules in parallel reveals which corrections belong to dependence and which belong to the score map.

The second-order conclusions are fixed-dimensional. The final block of [Appendix A](#) explains why fixed-dimensional MRV is insufficient when m grows, gives uniform-MRV projection and radial-kernel criteria, and proves dimension-uniform first-order validity for fixed-parameter positive Clayton and standard multivariate- t common-factor arrays. It also shows why dimension-free coefficients do not control second-order remainders. The main open problems are a recursive theorem for nested critical and nonintegrable faces, unconditional Gaussian boundary control, the angularly critical standard- t cases, and uniform control of active-face multiplicity when m grows.

More broadly, the classification applies to dependence structures rather than only named copula families. Organizing them by exponent-measure support and by how finite-threshold mass crosses the aggregation boundary turns universal first-order calibration into a quantitative theory of rate, coefficient, and sign, linking extreme-value geometry to finite-level heavily right combination tests.

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A Technical extensions and supporting results

This appendix collects results and extended derivations that support the main geometric and calibration arguments. All proof environments remain in the separate proofs appendix. Throughout, U_i denotes an exact-uniform p -value, $Y_i = U_i^{-1}$, $H_i = \cot(\pi U_i/2)$, $R_w = \sum_i w_i Y_i$, $T_w = \sum_i w_i H_i$, and $c = 2/\pi$.

A.1 Exponent measures, Poisson limits, and max-stability

The term *exponent measure* is most concrete through an extreme point process. If $\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(n)}$ are independent copies of an index-one MRV vector, then, under the usual point process formulation,

$$N_n = \sum_{k=1}^n \delta_{\mathbf{X}^{(k)}/n} \implies N, \quad (\text{A.1})$$

where N is a Poisson point process with intensity μ . Consequently, $\mu(A)$ is the limiting expected number, among n observations, whose normalized extreme vector falls in A . In particular,

$$\mathbb{P}\{N(A) = 0\} = \exp\{-\mu(A)\}. \quad (\text{A.2})$$

Let $\mathbf{M}_n$ denote the componentwise maximum of the n copies. For $\mathbf{x} > \mathbf{0}$, put

$$V(\mathbf{x}) = \mu([0, \infty]^m \setminus [\mathbf{0}, \mathbf{x}]) = \mu\{\mathbf{y} : y_i > x_i \text{ for some } i\}. \quad (\text{A.3})$$

Then

$$\begin{aligned} \mathbb{P}(\mathbf{M}_n/n \leq \mathbf{x}) &= [1 - \mathbb{P}\{X_i > nx_i \text{ for some } i\}]^n \\ &\longrightarrow \exp\{-V(\mathbf{x})\}. \end{aligned} \quad (\text{A.4})$$

The exponential therefore comes from the zero-count probability of a Poisson limit, equivalently from $(1 - V/n)^n \rightarrow e^{-V}$. This is the origin of the word “exponent”; it is unrelated to a Lie-group exponential map.

Homogeneity gives $V(a\mathbf{x}) = a^{-1}V(\mathbf{x})$, and hence

$$G(\mathbf{x}) = e^{-V(\mathbf{x})} \implies G(a\mathbf{x})^a = G(\mathbf{x})$$

for integer a , with the usual continuous extension. Thus the maximum of a independent vectors having distribution G , divided by a , has distribution G again. This is the meaning of *max-stable* at index one.

There is a limited Haar-measure analogy. Positive scalars act on $\mathbb{E}_m$ by dilation. Haar measure on the multiplicative group $(0, \infty)$ is dr/r , whereas the index- α radial exponent measure is

$$\alpha r^{-\alpha-1} dr = \alpha r^{-\alpha} \frac{dr}{r}. \quad (\text{A.5})$$

It is therefore Haar measure multiplied by the character $r^{-\alpha}$: the exponent measure is relatively invariant, not invariant, under dilations. In logarithmic radius $u = \log r$, the radial measure is the exponential tilt $\alpha e^{-\alpha u} du$ of translation-invariant Haar measure. This observation explains homogeneity, but not the terminology “exponent measure.”

The lower-tail replicate interpretation is the same construction after reciprocation. For independent copies $\mathbf{U}^{(1)}, \dots, \mathbf{U}^{(n)}$, set $M_{n,i} = \min_{1 \leq k \leq n} U_i^{(k)}$. If the lower stable-tail function ℓ_L in [Definition 3.1](#) exists, then

$$\mathbb{P}(nM_{n,i} > x_i \text{ for every } i) = \left[1 - \mathbb{P}\left\{ \bigcup_i (U_i \leq x_i/n) \right\} \right]^n \longrightarrow \exp\{-\ell_L(\mathbf{x})\}. \quad (\text{A.6})$$

Thus ℓ_L is the exponent of the limiting min-stable law. Since

$$\min_{1 \leq k \leq n} U_i^{(k)} = \left\{ \max_{1 \leq k \leq n} Y_i^{(k)} \right\}^{-1}, \quad (\text{A.7})$$

the same statement is equivalently a max-domain limit for $Y = 1/U$ and index-one MRV of Y . This also explains the phrase “lower max-domain of attraction.” In two dimensions, the union in [Eq. \(A.6\)](#) describes at least one small p -value, while the intersection probability $C(sx, sy)$ describes both being small; the limit $C(sx, sy)/s \rightarrow \Lambda_L(x, y)$ is the corresponding first-order overlap.

A.2 A copula outside the lower max-domain of attraction

The existence requirement in [Definition 3.1](#) is substantive. Uniform margins do not prevent the copula from alternating between different lower-tail clustering patterns on successively smaller scales. The following bivariate construction gives an explicit example.

Let $s_0 = e^{-2}$ and, for $0 < s \leq s_0$, define

$$\delta(s) = s \left[\frac{1}{2} + \frac{1}{8} \sin\{\log \log(1/s)\} \right], \quad \delta(0) = 0. \quad (\text{A.8})$$

Extend δ linearly from $(s_0, \delta(s_0))$ to $(1, 1)$:

$$\delta(s) = \delta(s_0) + \frac{1 - \delta(s_0)}{1 - s_0} (s - s_0), \quad s_0 \leq s \leq 1. \quad (\text{A.9})$$

On $(0, s_0]$,

$$\delta'(s) = \frac{1}{2} + \frac{1}{8} \sin\{\log \log(1/s)\} - \frac{\cos\{\log \log(1/s)\}}{8 \log(1/s)},$$

so $0 < \delta'(s) < 2$. The linear extension also has slope in $(0, 2)$, and the construction satisfies

$$\delta(1) = 1, \quad \delta(s) \leq s, \quad 0 \leq \delta(v) - \delta(u) \leq 2(v - u) \quad (0 \leq u \leq v \leq 1).$$

These are the standard conditions for a bivariate copula diagonal. Hence there exists a copula C_δ with $C_\delta(s, s) = \delta(s)$; diagonal-section characterizations and explicit constructions are given by Durante and Jaworski [\[3\]](#).

For exact uniform margins,

$$\frac{C_\delta(s, s)}{s} = \mathbb{P}(U_2 \leq s \mid U_1 \leq s) = \frac{1}{2} + \frac{1}{8} \sin\{\log \log(1/s)\}, \quad 0 < s \leq s_0. \quad (\text{A.10})$$

Along the sequences

$$\begin{aligned} s_n^{(+)} &= \exp \left[-\exp \left\{ \frac{\pi}{2} + 2\pi n \right\} \right], \\ s_n^{(-)} &= \exp \left[-\exp \left\{ \frac{3\pi}{2} + 2\pi n \right\} \right], \end{aligned}$$

the ratio in [Eq. \(A.10\)](#) equals $5/8$ and $3/8$, respectively. Therefore the lower-tail coefficient $\Lambda_L(1, 1)$ does not exist. Equivalently,

$$t\mathbb{P}(Y_1 > t, Y_2 > t) = tC_\delta(1/t, 1/t) = \frac{C_\delta(1/t, 1/t)}{1/t} \quad (\text{A.11})$$

has different subsequential limits, so (Y_1, Y_2) has no unique MRV tail measure.

The normalized lower-minimum probability also oscillates:

$$\frac{1}{s} \mathbb{P}\{\min(U_1, U_2) \leq s\} = 2 - \frac{C_\delta(s, s)}{s}, \quad (\text{A.12})$$

with subsequential values $11/8$ and $13/8$. The copula therefore has no stable first-order lower-extreme geometry, even though both marginal p -values are exactly uniform.

This example proves failure of lower MDA/MRV, not automatically failure of either aggregate-tail ratio. Convergence of an entire multivariate tail measure is stronger than convergence of one particular linear-functional tail; a reciprocal or positive Half-Cauchy weighted sum can conceivably converge even when the full measure does not. A counterexample to first-order aggregation validity itself would have to control the oscillating copula mass relative to the corresponding weighted-sum half-space, not merely its diagonal. Standard parametric copulas are usually much more regular; the example is intentionally scale-oscillatory.

A.3 Quantitative rates and a hierarchy of assumptions

It is important not to conflate three logically different conditions.

Level I: lower max-domain attraction, or ordinary MRV.

This is the existence of the first-order limit

$$t\mathbb{P}(X/t \in A) \longrightarrow \mu_X(A), \quad X \in \{Y, H\},$$

on fixed continuity sets. For the corresponding weighted half-space it gives only

$$\mathbb{P}\left(\sum_i w_i X_i > t\right) = \frac{c_X}{t} + o(t^{-1}), \quad c_Y = 1, \quad c_H = \frac{2}{\pi}.$$

It supplies no numerical rate for the little- o remainder.

Level II: quantitative second-order MRV.

This adds a rate to the convergence of the exponent measures on a class containing the weighted-sum half-space. It is the general condition needed for a second-order aggregate-tail conclusion and allows both asymptotic independence and asymptotic dependence.

Level III: product-type pair control.

A shell-probability bound is an elementary sufficient condition for the desired Level-II rate in the asymptotically independent case. A direct bound on each pairwise score density is stronger and implies the shell bound. A globally bounded bivariate copula density is an even simpler but still stricter way to verify the score-density bound. None of these Level-III conditions is necessary for Level II.

These levels form a one-way hierarchy. The examples and quantitative projection below clarify its strict steps; the full chain ends this section.

Example A.1 (Axial lower MDA with an arbitrarily slow, nonregular diagonal remainder). For s sufficiently small, put

$$r(s) = \log(1/s), \quad \eta(s) = \frac{2 + \sin r(s)}{\log r(s)}, \quad \delta(s) = s\eta(s). \quad (\text{A.13})$$

Choose $s_0 > 0$ small enough and extend δ linearly from $(s_0, \delta(s_0))$ to $(1, 1)$, with $\delta(0) = 0$. On $(0, s_0]$,

$$\delta'(s) = \frac{2 + \sin r - \cos r}{\log r} + \frac{2 + \sin r}{r(\log r)^2}.$$

For sufficiently small s_0 , this derivative lies in $(0, 2)$, $\delta(s) \leq s$, and the linear extension has the same properties. Thus δ is an admissible bivariate copula diagonal, so there is a copula C_δ satisfying $C_\delta(s, s) = \delta(s)$; see Durante and Jaworski [3].

This copula is nevertheless in the axial lower max-domain of attraction. Indeed, for fixed $x, y > 0$ and $M = \max(x, y)$, monotonicity gives

$$0 \leq \frac{C_\delta(sx, sy)}{s} \leq \frac{C_\delta(sM, sM)}{s} = M\eta(sM) \longrightarrow 0. \quad (\text{A.14})$$

Hence every pairwise comparable lower-tail event is $o(s)$ and Proposition 3.3 gives axis-supported index-one MRV.

No regularly varying normalization makes the diagonal remainder converge to a finite nonzero constant. Indeed, the quantity $C_\delta(s, s)/s = \eta(s)$ tends to zero more slowly than s^κ for every $\kappa > 0$, while

$$\frac{\eta(e^{-\pi}s)}{\eta(s)}$$

has subsequential limits $1/3$ and 3 . Thus ordinary lower MDA supplies neither a positive-power rate nor regular variation of its rectangle errors. This diagonal obstruction does not exclude a global second-order signed measure vanishing on this rectangle or a special cancellation on one weighted half-space.

A.4 Concrete conditions near independence

The following probability and density assumptions provide such control for copulas sufficiently close to product behavior in the relevant lower tail. The most general elementary version used here is a probability bound on joint shells; it does not require the pair to possess a joint density.

Proposition A.2 (Shell-probability alternative). *Let $\mathbf{w}$ be a fixed simplex weight vector and, after discarding zero weights, let m be the number of active coordinates. Let $\mathbf{X}$ be either $\mathbf{Y}$ or $\mathbf{H}$, put $c_Y = 1$, $c_H = 2/\pi$, and set $Z_i = w_i X_i$. Suppose that, for every ordered pair $i \neq j$, there is $K_{ij} < \infty$ such that, for all $0 \leq a < b < \infty$ and $y \geq 0$,*

$$\mathbb{P}(a < Z_i \leq b, Z_j > y) \leq \frac{K_{ij}}{1+y} \int_a^b \frac{dx}{(1+x)^2}. \quad (\text{A.15})$$

Then

$$\mathbb{P}\left(\sum_i w_i X_i > t\right) = \frac{c_X}{t} + O\left(\frac{\log t}{t^2}\right). \quad (\text{A.16})$$

The next density condition is stronger but often easier to verify.

Theorem A.3 (Product-type pair densities). *Let $\mathbf{w}$ be a fixed simplex weight vector and, after discarding zero weights, let m be the number of active coordinates. Let $\mathbf{X}$ be either $\mathbf{Y}$ or $\mathbf{H}$, put $c_Y = 1$, $c_H = 2/\pi$, and set $Z_i = w_i X_i$. Suppose every pair (Z_i, Z_j) is absolutely continuous and, for constants $K_{ij} < \infty$,*

$$g_{ij}(x, y) \leq \frac{K_{ij}}{(1+x)^2(1+y)^2}, \quad x, y \geq 0. \quad (\text{A.17})$$

Then Eq. (A.16) holds.

Condition Eq. (A.17) implies the shell-probability bound Eq. (A.15), because

$$\int_y^\infty \frac{dz}{(1+z)^2} = \frac{1}{1+y}.$$

Thus the shell-probability hypothesis in [Proposition A.2](#) is weaker than the product-density hypothesis in [Theorem A.3](#); the proposition is therefore the logically stronger statement and subsumes the theorem. The density formulation is retained because its proof makes the source of the logarithmic integral transparent.

Corollary A.4 (Bounded bivariate copula densities). *The conclusion of [Theorem A.3](#) holds for both score vectors if each bivariate p -value copula is absolutely continuous and*

$$\sup_{0 < u, v < 1} c_{ij}(u, v) < \infty. \quad (\text{A.18})$$

The global bound [Eq. \(A.18\)](#) is stronger than the proof needs. Since one selected score in the proof is at least t/m , it is enough that, for some $u_0 < 1$, the copula densities are bounded on both lower-tail strips

$$\sup_{0 < u \leq u_0, 0 < v < 1} c_{ij}(u, v) < \infty, \quad \sup_{0 < v \leq u_0, 0 < u < 1} c_{ij}(u, v) < \infty. \quad (\text{A.19})$$

Thus a density singularity confined to the upper–upper corner need not affect either aggregate’s upper tail. Conversely, bounded density implies

$$C_{ij}(s, s) \leq Ms^2,$$

so it excludes every copula with positive lower-tail-dependence coefficient. This explains why Level III cannot cover all Level-II MRV models.

A.5 General quantitative half-space projection

The preceding examples and sufficient conditions concern sharply different boundary behaviors. Their common abstract formulation is quantitative convergence of the exponent measure on the particular weighted half-space.

Proposition A.5 (Projection of a quantitative tail-measure bound). *Under the assumptions of [Theorem 2.3](#), let X be either Y or H , put $c_Y = 1$, $c_H = 2/\pi$, and let*

$$A_{\mathbf{w}} = \{\mathbf{x} : \mathbf{w}^\top \mathbf{x} > 1\}.$$

Suppose that, for some deterministic $a(t) \downarrow 0$,

$$t\mathbb{P}(X/t \in A_{\mathbf{w}}) = \mu_X(A_{\mathbf{w}}) + O\{a(t)\}. \quad (\text{A.20})$$

Then

$$\mathbb{P}\left(\sum_i w_i X_i > t\right) = \frac{c_X}{t} + O\left(\frac{a(t)}{t}\right). \quad (\text{A.21})$$

In particular, if

$$a(t) = O\left(\frac{\log t}{t}\right), \quad (\text{A.22})$$

then

$$\boxed{\mathbb{P}\left(\sum_i w_i X_i > t\right) = \frac{c_X}{t} + O\left(\frac{\log t}{t^2}\right)}. \quad (\text{A.23})$$

Condition [Eq. \(A.20\)](#) concerns the exponent measure on the relevant half-space, not merely the marginal tails. A conventional second-order MRV formulation assumes, on a suitable class of continuity sets,

$$t\mathbb{P}(\mathbf{X}/t \in \cdot) = \mu_X(\cdot) + A(t)\nu(\cdot) + o\{A(t)\}.$$

[Proposition A.5](#) follows when $A(t) = O\{(\log t)/t\}$, the weighted half-space belongs to the class on which second-order convergence is assumed, and $|\nu|(\partial A_w) = 0$. Finiteness of $\nu(A_w)$ alone is not a signed-measure portmanteau condition.

In lower-copula notation, a necessary diagnostic for such a uniform measure-level rate is

$$\frac{1}{s}\mathbb{P}\left(\bigcup_i \{p_i \leq sx_i\}\right) = \ell_L(\mathbf{x}) + O\{s \log(1/s)\}, \quad s \downarrow 0, \quad (\text{A.24})$$

uniformly on the required compact sets. Condition [Eq. \(A.24\)](#) on rectangle complements alone is not always sufficient for the weighted-sum half-space; a convergence-determining class and boundary control are also needed. It is, however, useful for ruling out the desired rate: if [Eq. \(A.24\)](#) already fails there, the full standard second-order exponent-measure condition fails.

The conclusions of this section are more accurately summarized by two one-way chains:

$$\begin{aligned} &\text{globally bounded bivariate copula densities} \\ &\implies \text{product-type pairwise score densities} \\ &\implies \text{joint shell-probability bounds} \\ &\implies \begin{aligned} &\text{quantitative aggregate-tail control on } A_w, \\ &\text{and axis-supported ordinary lower MDA/MRV;} \end{aligned} \quad (\text{A.25}) \\ &\text{second-order MRV on a convergence-determining class} \\ &\implies \begin{aligned} &\text{ordinary MRV, and quantitative control on } A_w \\ &\text{when that half-space is admissible.} \end{aligned} \end{aligned}$$

For the first chain, the shell bound gives pairwise comparable-exceedance probabilities of order $O(s^2)$ by taking one shell endpoint to infinity; [Proposition 3.3](#) then gives axial MRV. One-half-space control does not imply full MRV, and no reverse implication holds in general.

A.6 Exact marginal normalization on the axes

Proposition A.6 (Exact Pareto margins rule out a pure-axis second-order measure). *Let $\mathbf{Y}$ have exact unit-Pareto margins. Suppose, for a nonzero $A(t) \rightarrow 0$ and a signed Radon measure ν_2 , that*

$$t\mathbb{P}(\mathbf{Y}/t \in \cdot) = \mu(\cdot) + A(t)\nu_2(\cdot) + o\{A(t)\} \quad (\text{A.26})$$

holds on all marginal upper half-spaces $B_i(x) = \{\mathbf{y} : y_i > x\}$, $x > 0$. Then

$$\nu_2\{\mathbf{y} : y_i > x\} = 0 \quad (i = 1, \dots, m, x > 0). \quad (\text{A.27})$$

If ν_2 is supported on the union of the coordinate axes, then $\nu_2 = 0$.

For exact standard Half-Cauchy margins, the marginal expansion instead is

$$t\mathbb{P}(H_i > tx) = \frac{c}{x} - \frac{c}{3x^3}t^{-2} + O(t^{-4}), \quad c = \frac{2}{\pi}, \quad (\text{A.28})$$

so a nonzero pure-axis score-transformation correction exists at relative order t^{-2} , equivalently absolute tail order t^{-3} .

A.7 The product-power face trichotomy

Proposition A.7 (Symmetric product-power face trichotomy). *Let $w_1, w_2 > 0$ and let an open pair-face measure have density*

$$\nu_a(dx_1, dx_2) = Ka^2x_1^{-a-1}x_2^{-a-1}dx_1dx_2, \quad a > 0. \quad (\text{A.29})$$

Then the behavior of the absolute face integral near either axis is

$$\int_{\min(x_1, x_2) > \varepsilon} |G_{12}| d\nu_a = \begin{cases} O(1), & 0 < a < 1, \\ \Theta\{\log(1/\varepsilon)\}, & a = 1, \\ \Theta(\varepsilon^{1-a}), & a > 1, \end{cases} \quad \varepsilon \downarrow 0, \quad (\text{A.30})$$

up to finite contributions away from the axes. Thus $a < 1$, $a = 1$, and $a > 1$ yield integrability, criticality, and polynomial nonintegrability, respectively.

A.8 Support-agnostic common-factor inversion

On this index-one scale, the following support-agnostic theorem gives the reusable inversion. Its principal applications have interior mass, but it also covers the mixed one-sided multivariate- t model.

Theorem A.8 (Common-heavy-factor aggregation). *Let Q and Θ be independent. Suppose, for $a > 0$, $\rho > 0$, and $b \in \mathbb{R}$,*

$$\mathbb{P}(Q > q) = aq^{-1}\{1 - bq^{-\rho} + r_Q(q)\}, \quad r_Q(q) = o(q^{-\rho}). \quad (\text{A.31})$$

Let $S(q, \Theta)$ be a nonnegative statistic. For some finite collection $\lambda_j > 0$, suppose its large- q expansion is

$$S(q, \Theta) = qA(\Theta) \left\{ 1 + \sum_{j=1}^J c_j(\Theta)q^{-\lambda_j} + r_S(q, \Theta) \right\}. \quad (\text{A.32})$$

Put

$$\delta = \min\{\rho, \lambda_1, \dots, \lambda_J\}.$$

For large t , assume that the upper tail event can be written

$$\{S(Q, \Theta) > t\} = \{Q > q_t(\Theta)\}, \quad (\text{A.33})$$

up to an event of probability $o(t^{-1-\delta})$, where $q_t \rightarrow \infty$ almost surely on $\{A > 0\}$; set $q_t = \infty$ on $\{A = 0\}$. Assume the following explicit dominated-inversion conditions:

$$\mathbb{E} \left| q_t^{-1} - \frac{A}{t} - \sum_{j:\lambda_j=\delta} c_j A^{1+\delta} t^{-1-\delta} \right| = o(t^{-1-\delta}), \quad (\text{A.34})$$

$$\mathbb{E}(q_t^{-1-\rho}) = t^{-1-\rho} \mathbb{E}(A^{1+\rho}) + o(t^{-1-\delta}), \quad (\text{A.35})$$

$$\mathbb{E}\{q_t^{-1} | r_Q(q_t)\} = o(t^{-1-\delta}). \quad (\text{A.36})$$

Here and below a product containing A is defined as zero on $\{A = 0\}$. Then

$$\mathbb{P}\{S(Q, \Theta) > t\} = \frac{a\mathbb{E}A}{t} + K_\delta t^{-1-\delta} + o(t^{-1-\delta}), \quad (\text{A.37})$$

where

$$K_\delta = a \sum_{j:\lambda_j=\delta} \mathbb{E}(c_j A^{1+\delta}) - a \mathbf{1}\{\rho = \delta\} b \mathbb{E}(A^{1+\delta}). \quad (\text{A.38})$$

Thus corrections having the smallest score-scale exponent add, and a radial correction adds precisely when its exponent ties that minimum.

Remark (How to verify the inversion assumptions). On any angular truncation on which A is bounded away from zero and all coefficients are bounded, ordinary algebraic inversion of Eq. (A.32) gives

$$q_t^{-1} = \frac{A}{t} + \sum_{j: \mathcal{A}_j = \delta} c_j A^{1+\delta} t^{-1-\delta} + o(t^{-1-\delta}).$$

Eqs. (A.34) and (A.36) are exactly the uniform-integrability conditions needed to remove that truncation. They can be checked using moments of A , $c_j A^{1+\delta}$, and $A^{1+\rho}$ together with a negligible bound for the tail event outside the truncation. Exponential angular tails verify them for positive Clayton; Gaussian moment and local integrability bounds verify them for standard multivariate t when $\nu > 1$. This explicit formulation replaces the ambiguous phrase “assume dominated tail inversion.”

The theorem immediately separates bounded angular effects in reciprocal/harmonic-mean scores from the intrinsic third-order expansion of the positive Half-Cauchy map.

Corollary A.9 (Reciprocal/harmonic-mean common-factor phase diagram). *Suppose Eq. (A.31) holds and, under the dominated inversion conditions of Theorem A.8,*

$$S_Y(q, \Theta) = qA + q^{1-\kappa}B + C + o(q^{1-\kappa} + 1). \quad (\text{A.39})$$

Then, with $\delta = \min(\rho, \kappa, 1)$,

$$\mathbb{P}\{S_Y(Q, \Theta) > t\} = \frac{a\mathbb{E}A}{t} + K_Y t^{-1-\delta} + o(t^{-1-\delta}), \quad (\text{A.40})$$

$$K_Y = a\mathbf{1}\{\kappa = \delta\}\mathbb{E}(A^\kappa B) + a\mathbf{1}\{1 = \delta\}\mathbb{E}(AC) - ab\mathbf{1}\{\rho = \delta\}\mathbb{E}(A^{1+\rho}). \quad (\text{A.41})$$

Terms tie and add. In particular, when $\rho = \kappa$, the common-factor correction is $a\mathbb{E}\{A^\kappa(B - bA)\}$ provided $\rho = \kappa < 1$; at $\rho = \kappa = 1$ the bounded angular coefficient $a\mathbb{E}(AC)$ also ties. A nonzero bounded angular edge dominates the common-factor terms when $\rho = \kappa > 1$ provided its projected coefficient $\mathbb{E}(AC)$ does not cancel.

If $C \equiv 0$ and the sharper expansion

$$S_Y(q, \Theta) = qA + q^{1-\kappa}B + o(q^{1-\kappa}) \quad (\text{A.42})$$

holds under the dominated-inversion conditions of Theorem A.8 at $\delta_0 = \min(\rho, \kappa)$, then the exponent one must be omitted:

$$\mathbb{P}\{S_Y(Q, \Theta) > t\} = \frac{a\mathbb{E}A}{t} + K_0 t^{-1-\delta_0} + o(t^{-1-\delta_0}), \quad (\text{A.43})$$

$$K_0 = a\mathbf{1}\{\kappa = \delta_0\}\mathbb{E}(A^\kappa B) - ab\mathbf{1}\{\rho = \delta_0\}\mathbb{E}(A^{1+\rho}). \quad (\text{A.44})$$

In particular, if $\rho = \kappa$ at any positive value, then $K_0 = a\mathbb{E}\{A^\kappa(B - bA)\}$.

Corollary A.10 (Positive Half-Cauchy common-factor phase diagram). *Suppose Eq. (A.31) holds with $\rho = \kappa$ and, under the dominated inversion conditions of Theorem A.8,*

$$S_H(q, \Theta) = cqA + cq^{1-\kappa}B - dq^{-1}D + o(q^{1-\kappa} + q^{-1}), \quad c = \frac{2}{\pi}, \quad d = \frac{\pi}{6}. \quad (\text{A.45})$$

This hypothesis deliberately excludes a nonzero $O(1)$ angular term. If such a term is present, it enters at relative exponent one and must be included by applying Theorem A.8 directly. The applications below of this corollary have no such term. Then

$$\mathbb{P}\{S_H(Q, \Theta) > t\} = \frac{ac\mathbb{E}A}{t} + K_H t^{-1-\min(\kappa, 2)} + o\{t^{-1-\min(\kappa, 2)}\}, \quad (\text{A.46})$$

where

$$K_H = \begin{cases} ac^{1+\kappa}\mathbb{E}\{A^\kappa(B-bA)\}, & 0 < \kappa < 2, \\ ac^3\mathbb{E}\{A^2(B-bA)\} - adc^2\mathbb{E}(A^2D), & \kappa = 2, \\ -adc^2\mathbb{E}(A^2D), & \kappa > 2. \end{cases} \quad (\text{A.47})$$

Consequently, the dependence/radial perturbation dominates for $\kappa < 2$, collides with the Half-Cauchy transformation at $\kappa = 2$, and is hidden behind the universal t^{-3} transformation scale for $\kappa > 2$. The conclusion at $\kappa = 2$ requires $\mathbb{E}(A^2|D|) < \infty$; failure of this angular moment is a critical-face problem rather than a failure of the radial expansion.

A.9 Standard proper-face constructions and an exact max-linear example

Genuine proper faces first appear when $m \geq 3$. In two dimensions the simplex has only its two axis vertices and its open interior, so there is no intermediate coordinate-face dimension. Several standard multivariate extreme-value constructions can be parameterized to produce pure proper-face support:

1. *Marshall–Olkin and max-linear common-shock copulas* [17]. Independent shocks indexed by nonempty subsets $J \subseteq [m]$ generate spectral atoms on rays contained in F_J° . Shocks affecting pairs or larger proper blocks can therefore concentrate the exponent measure on selected proper faces. For small- p analysis one uses the survival/reflected orientation that maps large max-stable variables to small uniforms.
2. *Tawn asymmetric logistic extreme-value copulas* [18]. Their subset components can allocate diffuse spectral mass to selected coordinate faces. Retaining only components indexed by proper subsets produces a proper-face-supported model; adding axes or a full-set component moves the model into the mixed category below.

These examples concern support in the tail orientation relevant to the chosen extremes. Reflecting a copula can move upper-tail face structure to the lower tail of the p -values and must be stated explicitly.

A concrete three-dimensional max-linear example makes the geometry exact. Let Z_{12}, Z_{13}, Z_{23} be independent unit-Fréchet variables and set

$$X_1 = \max(Z_{12}/2, Z_{13}/2), \quad X_2 = \max(Z_{12}/2, Z_{23}/2), \quad X_3 = \max(Z_{13}/2, Z_{23}/2). \quad (\text{A.48})$$

Every margin is unit Fréchet and

$$S = \delta_{(1/2, 1/2, 0)} + \delta_{(1/2, 0, 1/2)} + \delta_{(0, 1/2, 1/2)}. \quad (\text{A.49})$$

Thus the first-order exponent measure is concentrated exactly on the three pair-edge rays. Pairwise extremes have order t^{-1} , while a three-way extreme requires at least two independent shocks and is of smaller order. The exact-uniform transformation

$$U_i = 1 - \exp(-1/X_i) \quad (\text{A.50})$$

makes small U_i correspond to large X_i , and $1/U_i \sim X_i$; hence the reciprocal score vector retains the same edge-supported exponent geometry. By [Proposition 2.1](#), the positive Half-Cauchy score vector has the identical support after the deterministic radial rescaling by c .

A.10 Three-edge critical integral

Lemma A.11 (Three-edge critical integral). *Let R_{12}, R_{13}, R_{23} be independent copies of a nonnegative continuous random variable with distribution function F_R and density f_R . Suppose, for some $a > 0$,*

$$1 - F_R(x) = \frac{a}{x} + O(x^{-2}), \quad f_R(x) = \frac{a}{x^2} + O(x^{-3}). \quad (\text{A.51})$$

For fixed positive weights satisfying $\sum_i w_i = 1$, put

$$G_{\mathbf{w}} = w_1 \max(R_{12}, R_{13}) + w_2 \max(R_{12}, R_{23}) + w_3 \max(R_{13}, R_{23}).$$

Then

$$\mathbb{P}(G_{\mathbf{w}} > t) = \frac{2a}{t} + 4a^2\sigma_2(\mathbf{w})\frac{\log t}{t^2} + O(t^{-2}). \quad (\text{A.52})$$

A.11 Higher-order Gaussian tails and the independence boundary

Ordinary MRV retains only the axis-supported first-order tail measure. It therefore discards the smaller probability that two Gaussian-copula scores are simultaneously large. Hidden regular variation recovers this probability. The following bivariate calculation identifies its order; see Fung and Seneta [5] and the multivariate heavy-tailed Gaussian-copula analysis of Das and Fasen-Hartmann [4].

Let (Z_1, Z_2) be standard bivariate normal with correlation $0 < \rho < 1$, put

$$U_i = 1 - \Phi(Z_i), \quad Y_i = U_i^{-1},$$

and define

$$q_\rho = \frac{2}{1 + \rho}, \quad a_\rho = \frac{1}{1 + \rho}.$$

Then, for an explicit constant $K_\rho > 0$,

$$\mathbb{P}(Y_1 > t, Y_2 > t) \sim K_\rho t^{-q_\rho} (\log t)^{-\rho/(1+\rho)}. \quad (\text{A.53})$$

Moreover, for fixed $x_1, x_2 > 0$, the corresponding hidden-tail ratio has the form

$$\frac{\mathbb{P}(Y_1 > tx_1, Y_2 > tx_2)}{\mathbb{P}(Y_1 > t, Y_2 > t)} \longrightarrow (x_1 x_2)^{-a_\rho}. \quad (\text{A.54})$$

Equivalently, when the deterministic scaling is taken to be $t^{-q_\rho} (\log t)^{-\rho/(1+\rho)}$, the hidden measure ν_{hid} has survival function $K_\rho (x_1 x_2)^{-a_\rho}$ and, in the interior quadrant, density proportional to

$$a_\rho^2 x_1^{-a_\rho-1} x_2^{-a_\rho-1}. \quad (\text{A.55})$$

Eqs. (A.54) and (A.55) identify the pair hidden measure. Its Möbius half-space increment is finite for every fixed $\rho > 0$, because $a_\rho < 1$. The conditional fixed-dimensional theorem below transfers this measure and sums all dominant pairs when explicit boundary and truncation conditions hold; those conditions are not verified here. Since $q_\rho < 2$, the hidden rectangle scale is larger than $(\log t)/t^2$ and already rules out the conventional Level-II measure-rate target. Under the stated transfer conditions it also becomes the aggregate correction recorded in Table 5.

It is tempting, but incorrect, to set $\rho = 0$ in the fixed- $\rho > 0$ formula and expect the independent answer. If $\rho = 0$, the comparable-extremes part of Eq. (A.53) is only of order t^{-2} . The larger independent remainder instead comes from edge regions in which one score is of order t while the other ranges over all intermediate scales:

$$\frac{1}{t^2} \int_1^t \frac{dy}{y} = \frac{\log t}{t^2}. \quad (\text{A.56})$$

The hidden-density calculation makes the nonuniformity explicit. Near an axis, the width of the weighted-sum shell in x_1 is proportional to x_2 . Combining this width with Eq. (A.55) produces an integral of the form

$$\int_0^1 y^{-a_\rho} dy = \frac{1}{1 - a_\rho} = \frac{1 + \rho}{\rho}, \quad \rho > 0. \quad (\text{A.57})$$

It is finite for every fixed positive ρ but diverges as $\rho \downarrow 0$. Keeping the finite- t cutoff gives the useful transition factor

$$\int_{1/t}^1 y^{-a_\rho} dy = \frac{1 - t^{-(1-a_\rho)}}{1 - a_\rho}. \quad (\text{A.58})$$

Consequently,

$$\frac{1 - t^{-(1-a_\rho)}}{1 - a_\rho} \sim \begin{cases} (1 - a_\rho)^{-1}, & (1 - a_\rho) \log t \rightarrow \infty, \\ \log t, & (1 - a_\rho) \log t \rightarrow 0. \end{cases} \quad (\text{A.59})$$

Since $1 - a_\rho \sim \rho$, the crossover occurs on the scale $\rho \log t = O(1)$. In particular,

$$\lim_{\rho \downarrow 0} \lim_{t \rightarrow \infty} \quad \text{and} \quad \lim_{t \rightarrow \infty} \lim_{\rho \downarrow 0}$$

are different asymptotic operations. The fixed- $\rho > 0$ expansion is not uniform at the independence boundary.

For one-sided tests with $-1 < \rho < 0$, the comparable lower-lower exponent $2/(1 + \rho)$ exceeds two, so that particular joint-extreme term is smaller than t^{-2} . It does not by itself identify the leading weighted-sum remainder, which may instead be governed by an edge contribution and needs a separate calculation. For two-sided tests, the four sign quadrants include a dominant positively correlated quadrant, and the corresponding pairwise exponent uses $|\rho|$:

$$q_{|\rho|} = \frac{2}{1 + |\rho|} < 2 \quad (\rho \neq 0).$$

Remark (Relation to the Gaussian-copula literature). The bivariate normal-copula regular-variation input is classical; the general cone geometry is part of the hidden-regular-variation literature for Gaussian-copula risks. The calculation here identifies the explicit face-increment coefficient for the weighted reciprocal/harmonic-mean half-space, the sum over all maximally correlated pairs, and the corresponding positive Half-Cauchy transfer under stated boundary conditions. This is more specialized than general Gaussian portfolio-tail theory and does not discover Gaussian hidden regular variation.

Open problem A.12 (One-sided nonpositive Gaussian dependence). *Suppose every active one-sided correlation is nonpositive and at least one is strictly negative. Then the positive-pair power term in [Theorem A.13](#) is absent. For both R_w and T_w the leading dependence correction can be an edge term, with independence as a critical logarithmic boundary; T_w also has its intrinsic score-map scale. A uniform theorem covering mixed zero and negative correlations remains to be written. The all-zero case is the product copula and is already settled by [Theorem 4.2](#).*

A.12 Conditional fixed-dimensional Gaussian expansion

The preceding bivariate calculation identifies the candidate pair measures and coefficients, but an arbitrary-dimensional weighted-half-space theorem also needs uniform control near every proper subface. The next result states that missing boundary requirement explicitly. Let $A = \{i : w_i > 0\}$ and, for one-sided tests, define

$$\rho_* = \max_{i < j, i, j \in A} \rho_{ij}, \quad E_* = \{(i, j) : i < j, i, j \in A, \rho_{ij} = \rho_*\}.$$

For $0 < \rho < 1$, put

$$a_\rho = \frac{1}{1 + \rho}, \quad q_\rho = 2a_\rho, \quad \beta_\rho = 1 - a_\rho, \quad (\text{A.60})$$

$$K_\rho = \frac{(1 + \rho)^2}{\sqrt{1 - \rho^2}} (4\pi)^{a_\rho - 1}, \quad C(a) = -\frac{\Gamma(1 - a)^2}{\Gamma(1 - 2a)}. \quad (\text{A.61})$$

Notice that $C(a) > 0$ for $1/2 < a < 1$.

Theorem A.13 (Gaussian aggregation under face-transfer conditions). *Let $m < \infty$, let $R = (\rho_{ij})$ be a positive-definite correlation matrix, and let $Y_i = \{1 - \Phi(Z_i)\}^{-1}$ for $\mathbf{Z} \sim N_m(0, R)$. Suppose $\rho_* > 0$ and put*

$$r_*(t) = t^{-q_{\rho_*}} (\log t)^{-\beta_{\rho_*}}.$$

Assume that every $(i, j) \in E_$ satisfies the uniform-integrability condition Eq. (5.4) at scale $r_*(t)$, and that the remaining Möbius faces satisfy*

$$\sum_{\substack{i < j \\ (i, j) \notin E_*}} |\mathbb{E} G_{\{i, j\}}(Y_i/t, Y_j/t)| + \sum_{\substack{I \subseteq A \\ |I| \geq 3}} |\mathbb{E} G_I(\mathbf{Y}_I/t)| = o\{r_*(t)\}. \quad (\text{A.62})$$

Then

$$\mathbb{P}\left(\sum_{i=1}^m w_i Y_i > t\right) = \frac{1}{t} + D_{G, \mathbf{w}} t^{-q_{\rho_*}} (\log t)^{-\beta_{\rho_*}} + o\left(t^{-q_{\rho_*}} (\log t)^{-\beta_{\rho_*}}\right), \quad (\text{A.63})$$

$$D_{G, \mathbf{w}} = K_{\rho_*} C(a_{\rho_*}) \sum_{(i, j) \in E_*} (w_i w_j)^{a_{\rho_*}}. \quad (\text{A.64})$$

For the Half-Cauchy scores and $c = 2/\pi$,

$$\mathbb{P}\left(\sum_i w_i H_i > t\right) = \frac{c}{t} + c^{q_{\rho_*}} D_{G, \mathbf{w}} t^{-q_{\rho_*}} (\log t)^{-\beta_{\rho_*}} + o\left(t^{-q_{\rho_*}} (\log t)^{-\beta_{\rho_*}}\right). \quad (\text{A.65})$$

For two-sided p -values $U_i = 2\{1 - \Phi(|Z_i|)\}$, replace ρ_ by*

$$\rho_*^{\text{abs}} = \max_{i < j, i, j \in A} |\rho_{ij}|,$$

replace E_ by the pairs attaining that maximum, and replace K_{ρ_*} by $2^{1-q_{\rho_*}} K_{\rho_*}$. The conclusions hold when $\rho_*^{\text{abs}} > 0$ and the analogous two-sided conditions hold.*

Remark (Status of the Gaussian boundary conditions). The bivariate normal tail calculation proves the hidden vague limit on compact subsets of the open pair face. What is not proved here is the uniform truncation at the adjacent axes and, in dimension at least three, the full estimate Eq. (A.62). A complete unconditional fixed- m theorem requires a stratified Gaussian boundary lemma controlling dominant pair shells, nonpositive pairs, and all higher Möbius faces. The displayed coefficient is therefore a conditional weighted-half-space conclusion, not an unconditional Gaussian model theorem.

The expansion is also pointwise in the correlation parameters. It is not uniform as $\rho_* \uparrow 1$: then $q_{\rho_*} \downarrow 1$, and the formal relative correction approaches $(\log t)^{-1/2}$. For every fixed $\rho_* < 1$ the relative correction remains polynomial, with exponent tending to zero as $\rho_* \uparrow 1$; only the boundary scale is logarithmic.

The bivariate hidden regular variation used above is classical. The contribution here is the weighted-half-space face coefficient and its positive Half-Cauchy transfer under explicit boundary hypotheses; the unresolved boundary conditions are stated explicitly above.

A.13 First-order multivariate- t copula geometry

Let $\mathbf{Z}$ have a standard m -variate t distribution with common degrees of freedom $\nu > 0$ and correlation matrix R . Use the stochastic representation

$$\mathbf{Z} = R_0 \mathbf{G}, \quad R_0 = \sqrt{\frac{\nu}{W}}, \quad (\text{A.66})$$

where $\mathbf{G} \sim N_m(\mathbf{0}, R)$, $W \sim \chi_\nu^2$, and $\mathbf{G}$ and W are independent.

Lemma A.14 (Regular variation of the multivariate t vector). *The random vector $\mathbf{Z}$ in Eq. (A.66) is multivariate regularly varying with index ν .*

Let T_ν denote the univariate t_ν CDF. Its upper tail satisfies

$$\bar{T}_\nu(z) = 1 - T_\nu(z) \sim k_\nu z^{-\nu}, \quad z \rightarrow \infty, \quad (\text{A.67})$$

for a constant $k_\nu > 0$.

Theorem A.15 (Multivariate- t -copula score vector). *For the one-sided p -values*

$$U_i = 1 - T_\nu(Z_i),$$

the vectors $(1/U_1, \dots, 1/U_m)$ and

$$(\cot(\pi U_1/2), \dots, \cot(\pi U_m/2))$$

are multivariate regularly varying with index one. The same conclusion holds for the two-sided p -values

$$U_i = 2\{1 - T_\nu(|Z_i|)\}.$$

Consequently, Theorem 2.3 applies to both R_w and T_w in both cases.

Unlike the Gaussian case, a multivariate- t copula has positive tail dependence. In the bivariate case its tail-dependence coefficient is

$$\lambda_t = 2T_{\nu+1}\left(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}}\right) > 0, \quad -1 < \rho < 1; \quad (\text{A.68})$$

see Demarta and McNeil [2]. Its limiting spectral measure consequently has interior mass. This is why the multivariate- t case can satisfy the MRV theorem while failing a pairwise $o(s)$ tail-independence condition.

Remark (Scope of the t proof). Theorem A.15 concerns the standard multivariate- t model with a common random scale, equivalently the usual multivariate- t copula. It does not automatically cover a collection of marginal Student statistics having different, data-dependent denominators. The joint copula of those statistics must be analyzed separately.

A.14 Radial source of the standard- t second-order rate

The standard multivariate- t copula is asymptotically dependent, so its higher-order mechanism is not Gaussian hidden regular variation. It comes from second-order regular variation of the common radial multiplier in Eq. (A.66), together with the second-order accuracy of the marginal score transformation. General second-order elliptical MRV and the multivariate- t example are treated by Kim [10].

The lower-tail expansion of the χ_ν^2 distribution gives, directly,

$$\mathbb{P}(R_0 > r) = \frac{\nu^{\nu/2}}{2^{\nu/2}\Gamma(\nu/2 + 1)} r^{-\nu} \left\{ 1 - \frac{\nu^2}{2(\nu+2)} r^{-2} + O(r^{-4}) \right\}. \quad (\text{A.69})$$

The leading score map is homogeneous of degree ν by Theorem A.15. Therefore a score threshold of order t corresponds to a radial threshold of order $t^{1/\nu}$, and the r^{-2} relative correction in Eq. (A.69) becomes the score-scale exponent-measure rate

$$A(t) \asymp t^{-2/\nu}. \quad (\text{A.70})$$

The exact t -CDF transformation and the signed Gaussian angles also contribute. Exact Pareto margins remove pure one-coordinate corrections, but they do not remove an interior perturbation of the exponent measure. The theorem in Theorem 8.1 performs the required inversion and shows how the radial scale competes with the bounded angular term at $\nu = 2$ and with the Half-Cauchy score-map scale at $\nu = 1$.

A.15 Coefficient control and the angularly critical standard- t boundary

Proposition A.16 (Dimension-free coefficient control). *For fixed $\nu > 1$, the absolute values of $D_{\nu, \mathbf{w}}^+$ and $D_{\nu, \mathbf{w}}^{|\cdot|}$ are bounded by a constant depending only on ν , uniformly over m , all correlation matrices, and all simplex weights. The additional one-sided angular coefficient $\mathbb{E}(A_+ C_-)$ is bounded uniformly as well.*

Remark (The unresolved critical and subcritical cases). At $\nu = 1$, $|G_i|^{\nu-2}$ is at the angular integrability boundary and the radial correction meets the t^{-3} Half-Cauchy term. For $0 < \nu < 1$, proper angular subfaces can dominate. These cases require the recursive nested-face theory discussed in [Section 9](#); no theorem for them is claimed here.

Remark (Fixed-dimensional scope). The expansions in this subsection are fixed-dimensional and concern $t \rightarrow \infty$, equivalently $\alpha \downarrow 0$. [Proposition A.16](#) controls their displayed coefficients, not their remainders, and therefore does not provide the uniformity needed for growing m .

A.16 Clayton versus multivariate t : the broader class

Both models are concrete instances of [Theorem A.8](#) after reciprocal score transformation. Their angular laws and second-order parameters differ: Thus [Table 4](#) shows that $\theta = 2/\nu$ is an equal-rate surface, not an

Table 4: Common-factor comparison of positive Clayton and standard multivariate- t copulas.

	positive Clayton	standard multivariate t_ν
score-scale common factor	$V^{-1/\theta}$	$(\nu/W)^{1/2}$
score homogeneity	one after frailty rewrite	degree ν
transformed tail index	one	one
score-scale second-order κ	θ	$2/\nu$
angular variables	positive exponentials	signed Gaussian powers
first-order spectral mass	interior	two-sided interior; one-sided mixed
angular critical boundary	none for first correction	$\nu = 1$

equality of copula families. Across the broader common-factor class, $\kappa < 2$ means the dependence/radial correction dominates the intrinsic t^{-3} Half-Cauchy term, $\kappa = 2$ is the collision point, and $\kappa > 2$ makes the Half-Cauchy transform dominate. This gives the Clayton transition $\theta = 2$ and the formal t transition $\nu = 1$. The latter is more singular because the Gaussian angular inverse moments also become critical. Thus [Corollary A.10](#) is a genuine shared theorem, while the verification of its dominated-inversion assumptions remains model-specific.

The class overlaps with Gaussian hidden regular variation only at the level of a generic face expansion, not at the first-order geometry. Gaussian copulas with fixed correlations strictly below one have axis-supported first-order MRV; their first dependence correction lives on hidden pair faces. Positive Clayton and two-sided t scores are interior-supported. One-sided t scores are mixed-supported but still have interior mass. In both t orientations, simultaneous extremes are already present at first order, so their next common-factor term perturbs the existing measure rather than marking the first appearance of an interior face.

Proposition A.17 (Uniform Clayton coefficient bounds). *For fixed $\theta > 0$,*

$$0 \leq D_{\theta, \mathbf{w}} \leq 1$$

uniformly over m and all simplex weights. For fixed $\theta > 1$, the same uniform bound holds for $\mathbb{E}(A_{\mathbf{w}}^2 Q_{\mathbf{w}})$.

A.17 Why fixed-dimensional MRV does not solve growing m

Definition 2.2 is posed on a fixed state space $\mathbb{E}_m$. When $m = m(t) \rightarrow \infty$, both the state space and the dependence law change with t . Row-by-row MRV gives a limit only after holding the row dimension fixed; it gives no uniform control at the diagonal point where the tail threshold and dimension grow together.

For the existing one-big-jump proof, the relevant remainder has the form

$$m_t^2 R_t, \quad R_t = \max_{i \neq j} \mathbb{P} \left(p_{i,t} < \frac{2w_{i,t}m_t}{\pi t}, p_{j,t} < \frac{2w_{j,t}m_t}{\pi \delta_t t} \right). \quad (\text{A.71})$$

It must be shown uniformly that

$$m_t^2 R_t = o(t^{-1}). \quad (\text{A.72})$$

A pointwise $o(t^{-1})$ statement for each pair cannot be summed over a growing collection of pairs. Likewise, for $\mathbf{X} \in \{\mathbf{Y}, \mathbf{H}\}$, fixed-dimensional convergence

$$t\mathbb{P}(\mathbf{X}/t \in A) \rightarrow \mu_{\mathbf{X}}(A)$$

does not provide a convergence rate uniform in m or in the weight vector.

A genuine growing-dimensional MRV theory needs a triangular-array condition that controls the tail-measure error uniformly over the moving half-spaces

$$\{\mathbf{x} : \mathbf{w}_t^\top \mathbf{x} > 1\}.$$

This is additional information, not a consequence of row-by-row MRV.

Liu, Meng and Pillai [33] give a rigorous first-order one-big-jump route for both positive Half-Cauchy and reciprocal aggregation. Its master assumption is Eq. (A.72), together with $\delta_t \downarrow 0$ and $t\delta_t \rightarrow \infty$. This proves calibration against the single Half-Cauchy or Pareto tail. The further comparison with the exact finite- m independence reference uses

$$\max_i w_{i,t} = O(m_t^{-1}), \quad m_t = o(\sqrt{t}).$$

A polynomial corollary follows if the pair event is $o\{t^{-(1+\gamma)}\}$ uniformly and $m_t = O(t^{\gamma/2})$. For pairwise Gaussian p -values with $|\rho_{ij}| \leq \rho_{\max} < 1$, put

$$\gamma_0 = \frac{1 - \rho_{\max}}{1 + \rho_{\max}}.$$

The proved general boundary is $m_t = o(t^{\gamma_0/2})$; when $0 < \rho_{\max} < 1$, the Gaussian logarithmic factor sharpens this to $m_t = O(t^{\gamma_0/2})$. If $\rho_{\max} = 0$ and only pairwise Gaussianity is assumed, the elementary argument retains the strict endpoint $m_t = o(\sqrt{t})$.

Those one-big-jump results are sufficient, not necessary. They are first-order and pair-accumulation based; they are naturally adapted to axis-supported arrays and exclude important interior-supported examples such as standard multivariate t , even though those examples can remain calibrated for arbitrary dimension by a common-factor argument.

A.18 A precise uniform-MRV theorem for growing dimension

Growing-dimensional results require explicit uniformity. The next theorem states exactly what is needed.

Theorem A.18 (Triangular-array aggregation under uniform MRV). Choose $X \in \{Y, H\}$, put $c_Y = 1$ and $c_H = 2/\pi$, and let $t_n \rightarrow \infty$ and $m_n \geq 1$. For row n , let $\mathbf{X}_n \in [0, \infty)^{m_n}$ be the reciprocal score vector with exact unit-Pareto margins when $X = Y$, or the positive Half-Cauchy score vector when $X = H$. Let $\mathbf{w}_n$ be nonnegative with $\sum_{i=1}^{m_n} w_{i,n} = 1$. Suppose there is an index-one homogeneous Radon measure μ_n on $\mathbb{E}_{m_n}$ satisfying

$$\mu_n\{\mathbf{x} : x_i > 1\} = c_X \quad (i = 1, \dots, m_n). \quad (\text{A.73})$$

Define the moving half-space

$$A_n = \{\mathbf{x} : \mathbf{w}_n^\top \mathbf{x} > 1\}.$$

If the triangular array satisfies the uniform tail approximation

$$|t_n \mathbb{P}(\mathbf{X}_n/t_n \in A_n) - \mu_n(A_n)| \rightarrow 0, \quad (\text{A.74})$$

then

$$t_n \mathbb{P}(\mathbf{w}_n^\top \mathbf{X}_n > t_n) \rightarrow c_X. \quad (\text{A.75})$$

There is no intrinsic restriction on the numerical growth rate of m_n ; the permissible rate is determined by the uniform approximation Eq. (A.74).

A.19 Metrics and radial kernels that verify uniform MRV

The half-space assumption in Theorem A.18 can be checked either by a measure metric or, more sharply for common-factor models, by a one-dimensional conditional radial tail.

Theorem A.19 (Bounded-Lipschitz control of moving half-spaces). Let $\nu_{n,t} = t\mathbb{P}(\mathbf{X}_n/t \in \cdot)$ and let μ_n be an index-one homogeneous measure with marginal constants c . Suppose that, for functions supported on $\{\|\mathbf{x}\|_\infty \geq 1/2\}$,

$$\left| \int f d(\nu_{n,t} - \mu_n) \right| \leq \delta_n(t) \{\|f\|_\infty + \text{Lip}(f)\}. \quad (\text{A.76})$$

Then, for $0 < \eta \leq 1/2$,

$$\sup_{\mathbf{w} \in \Delta_{m_n-1}} |t\mathbb{P}(\mathbf{w}^\top \mathbf{X}_n > t) - c| \leq \delta_n(t)(1 + \eta^{-1}) + \frac{c\eta}{1 - \eta}. \quad (\text{A.77})$$

In particular, choosing $\eta = \sqrt{\delta_n(t)}$ proves uniform weighted projection convergence whenever $\delta_n(t) \rightarrow 0$.

Theorem A.20 (Exact radial-angular error identity). For row n , suppose $\mathbf{X}_n = R_n \boldsymbol{\Theta}_n$, where $\boldsymbol{\Theta}_n$ lies on the positive ℓ_1 simplex and is independent of R_n . Assume, for r large,

$$\mathbb{P}(R_n > r) = \frac{C_n}{r} \{1 + e_n(r)\}, \quad C_n \mathbb{E} \boldsymbol{\Theta}_{i,n} = c \quad \text{for every } i.$$

Writing $L_{n,\mathbf{w}} = \mathbf{w}^\top \boldsymbol{\Theta}_n$, one has, for every sufficiently large t , the exact identity

$$t\mathbb{P}(\mathbf{w}^\top \mathbf{X}_n > t) - c = C_n \mathbb{E} [L_{n,\mathbf{w}} e_n\{t/L_{n,\mathbf{w}}\}]. \quad (\text{A.78})$$

The integrand is defined to be zero on $\{L_{n,\mathbf{w}} = 0\}$. Hence $\sup_{r \geq t} |e_n(r)| \rightarrow 0$ is sufficient, uniformly in dimension and weights.

More generally, independence may be replaced by the conditional kernel

$$\mathbb{P}(R_n > r \mid \boldsymbol{\Theta}_n = \mathbf{s}) = \frac{k_n(\mathbf{s})}{r} \{1 + e_n(r, \mathbf{s})\}, \quad \mathbb{E}\{k_n(\boldsymbol{\Theta}_n) \boldsymbol{\Theta}_{i,n}\} = c.$$

Then the right side of Eq. (A.78) becomes

$$\mathbb{E} [k_n(\boldsymbol{\Theta}_n) L_{n,\mathbf{w}} e_n\{t/L_{n,\mathbf{w}}, \boldsymbol{\Theta}_n\}].$$

A.20 Arbitrary-dimensional first-order theorems for the common-factor families

The only uniform product-tail input needed below is recorded explicitly. This avoids inferring uniformity from the ordinary fixed-law form of Breiman's lemma.

Lemma A.21 (Uniform index-one product tail). *Let $Q \geq 0$ satisfy*

$$\mathbb{P}(Q > q) = aq^{-1}\{1 + o(1)\}, \quad q \rightarrow \infty,$$

and let $S_n \geq 0$ be independent of Q . If, for some $\delta > 0$,

$$\sup_n \mathbb{E} S_n^{1+\delta} < \infty,$$

then, for every sequence $t_n \rightarrow \infty$,

$$t_n \mathbb{P}(QS_n > t_n) - a \mathbb{E} S_n \rightarrow 0. \quad (\text{A.79})$$

The conclusion remains valid for triangular arrays; the law and dimension of S_n may vary.

Theorem A.22 (Growing-dimensional standard multivariate- t aggregation). *Fix $\nu > 0$. Let m_n be arbitrary, let R_n be arbitrary m_n -dimensional correlation matrices, let $\mathbf{Z}_n \sim t_\nu(0, R_n)$ be the standard common-scale multivariate- t vector, and let $\mathbf{w}_n$ be arbitrary simplex weights. For either one- or two-sided exact t_ν p -values and any $t_n \rightarrow \infty$,*

$$t_n \mathbb{P}\left(\sum_i w_{i,n} Y_{i,n} > t_n\right) \rightarrow 1, \quad (\text{A.80})$$

$$t_n \mathbb{P}\left(\sum_i w_{i,n} H_{i,n} > t_n\right) \rightarrow \frac{2}{\pi}. \quad (\text{A.81})$$

There is no restriction on the rate at which m_n grows.

Theorem A.23 (Growing-dimensional positive Clayton aggregation). *Fix $\theta > 0$. For arbitrary m_n , arbitrary simplex weights, and m_n -variate Clayton copulas with the common fixed parameter θ , the conclusions Eqs. (A.80) and (A.81) hold for every $t_n \rightarrow \infty$, again with no restriction on the growth of m_n .*

These two theorems are MRV-first generalizations beyond a pairwise tail-independence argument: they allow interior spectral mass and thus cover genuine tail dependence. They complement, rather than duplicate, the axis-oriented one-big-jump theory of Liu, Meng and Pillai [33].

Conjecture A.24 (Uniform second-order common-factor expansions). *For fixed $\theta > 0$, the Clayton expansions in Theorems 8.2 and 8.3 should hold uniformly over m and simplex weights after their remainders are expressed through uniform exponential truncation bounds. For fixed $\nu > 1$, the same should hold for the standard multivariate- t expansions uniformly over m , weights, and a suitably nondegenerate class of correlation matrices. The bounds in Propositions A.16 and A.17 show that the displayed coefficients do not grow with dimension; they do not, by themselves, prove uniformity of the remainders.*

Open problem A.25 (Growing-dimensional Gaussian hidden faces). *Theorem A.13 is fixed-dimensional. A growing version must control uniformly: all pair-tail approximations, accumulation of triple and higher faces, covariance matrices approaching singularity, and the independence boundary $\rho \log t = O(1)$. As a diagnostic only, if all active maximal correlations equal a fixed $\rho > 0$ and the weights are of order $1/m$, the displayed pair coefficient can be of order $m^{2\rho/(1+\rho)}$. Ignoring the still-unproved uniform remainder would suggest*

$$m = o\left(t^{(1-\rho)/(2\rho)} \sqrt{\log t}\right).$$

This is not a theorem: higher-face multiplicities and uniform Gaussian tail errors must first be bounded.

A.21 Scope of the growing-dimensional results

The full one-big-jump development of Liu, Meng and Pillai [33] answers a different question under a different sufficient condition. The concise comparison above is included to explain why fixed-dimensional MRV cannot simply be diagonalized in m .

The two common-factor theorems show that interior spectral mass does not obstruct growing dimension and that a radial representation can be sharper than pair counting. The abstract uniform-MRV, bounded-Lipschitz, and uniform product-tail results make the required uniformity explicit; they are supporting tools for these first-order extensions rather than second-order calibration theorems.

No growing-dimensional second-order theorem is currently proved. The dimension-free t and Clayton coefficient bounds do not control their remainders, and the Gaussian polynomial rate in [Open problem A.25](#) is only a diagnostic. Accordingly, growing m should be presented as a first-order extension and a second-order open problem, not as part of the main second-order claim.

B Systematic copula classification and supporting proofs

This appendix contains the full copula ledger and verifies its support, rectangle-rate, and bounded-density assertions. Proved aggregate rates are either established in the main model theorems or derived explicitly below; the table cross-references those proofs. Mechanism statements explicitly marked “candidate,” “diagnostic,” or “open” are classifications suggested by the available tail geometry, not weighted-half-space theorems. Except where a multivariate model is explicitly stated, the calculations are bivariate. Specifying every bivariate margin of an m -dimensional copula does not in general determine its full lower stable-tail-dependence function when $m > 2$.

The proof-status language in the rate and mechanism columns is used literally. When a row says that an aggregate rate is proved, a full weighted-half-space conclusion has been established at the scale displayed in that row. In rows displaying the Level-II target, this means, for $X \in \{Y, H\}$ with $c_Y = 1$ and $c_H = 2/\pi$,

$$\mathbb{P}\left(\sum_i w_i X_i > t\right) = \frac{c_X}{t} + O\left(\frac{\log t}{t^2}\right).$$

Any different displayed power or slowly varying scale is to be read literally. A rectangle obstruction or a statement labelled “expected” is only diagnostic and is not promoted to a half-space theorem.

Table 5 classifies the families discussed in this paper. The orientation is always that small p -values are the relevant lower tail. The second column gives the four support categories; a diagonal ray lies in the full-simplex interior, whereas “proper face” is reserved for a coordinate face. The third column gives the absolute aggregate-tail rate after the universal t^{-1} term. The fourth separates proved half-space expansions from mechanism predictions based only on rectangle or hidden-tail diagnostics; each row states explicitly whether a sharp coefficient, only a rate bound, or merely a mechanism diagnosis is available. Standard formulas for these families can be found in Nelsen [8]. Corner-unbounded densities in several standard families, and the broader fact that smooth copulas need not have bounded densities, are discussed by Lalancette and Zimmerman [9].

Here “rate” means the absolute aggregate-tail correction beyond c_S/t , not the independence-relative distortion. A candidate rectangle or hidden scale in a row marked diagnostic or open is not asserted to be a weighted-half-space rate, and a displayed expansion scale can vanish for special weights or parameters.

Table 5: Exponent-measure support, aggregate second-order rate, mechanism, proof status, and bounded-density diagnostics.

Copula and parameters	pa- First-order support	Aggregate order rate	second- Mechanism and current status	Global bounded density
Product (independence)	(indep- Axes.	$t^{-2} \log t$ for both aggregates; sharp and proved.	Critical hidden pair faces; the sharp log-arithmetic coefficient is proved.	Yes, $c(u, v) = 1$.

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Copula and parameters	pa- First-order support	Aggregate order rate	second- Mechanism and current status	Global bounded density
FGM, $-1 \leq \theta \leq 1$	Axes.	$t^{-2} \log t$ for both aggregates when $\theta > -1$; $O(t^{-2})$ at $\theta = -1$; proved under the stated fixed-dimensional conditions.	Theorem 5.5 gives the critical coefficient from $\lambda = 1 + \theta$; at $\theta = -1$ the logarithmic coefficient vanishes and the remainder is $O(t^{-2})$.	Yes; $c(u, v) = 1 + \theta(1 - 2u)(1 - 2v)$.
AMH, $-1 \leq \theta < 1$	Axes.	$t^{-2} \log t$ for both aggregates; sharp bivariate and proved in higher dimension under the local-corner theorem's higher-face condition.	The sharp local-corner theorem applies with $\lambda = (1 - \theta)^{-1}$ and gives the logarithmic coefficient.	Yes for each fixed $\theta < 1$; the bound deteriorates as $\theta \uparrow 1$.
AMH, $\theta = 1$	Full-interior mass, with $\Lambda_L(x, y) = xy/(x + y)$.	t^{-2} for both aggregates; sharp and proved for two positive weights.	This endpoint is exactly bivariate Clayton with parameter one. The ordinary common-frailty/interior perturbation in Theorems 8.2 and 8.3 gives coefficients $w(1 - w)$ and $c^2 w(1 - w)$ for reciprocal and Half-Cauchy aggregation.	No; the density is unbounded at the lower corner.
Frank, finite fixed parameter	Axes.	$t^{-2} \log t$ for both aggregates; sharp bivariate and proved in higher dimension under the local-corner theorem's higher-face condition.	The sharp local-corner theorem applies with $\lambda = \theta/(1 - e^{-\theta})$, continuously interpreted as one at $\theta = 0$.	Yes; include independence as the zero-parameter limit.
Plackett, $0 < \theta < \infty$ fixed	Axes.	$t^{-2} \log t$ for both aggregates; sharp bivariate and proved in higher dimension under the local-corner theorem's higher-face condition.	The sharp local-corner theorem applies with $\lambda = \theta$ and gives the critical logarithmic coefficient.	Yes; the bound need not be uniform as the parameter degenerates.
Finite-degree Bernstein, finite checkerboard, and finite mixtures of bounded densities	Axes.	$O(t^{-2} \log t)$ for both aggregates; a sharp coefficient follows when the local-corner hypotheses hold.	The sharp local-corner theorem gives the coefficient whenever every pair density has a Dini-continuous corner limit and higher faces satisfy its bounded-strip condition; otherwise only the rate is asserted.	Yes, provided the degree, grid, and component bounds are fixed.

Continued on next page

Copula and parameters	First-order support	Aggregate order rate	second-order rate	Mechanism and current status	Global bounded density
One-sided Gaussian, $\rho = 0$	Axes; independence.	$t^{-2} \log t$ for both aggregates; sharp and proved.		Critical hidden pair faces; the sharp independent logarithm is proved.	Yes.
One-sided Gaussian, $0 < \rho < 1$	Axes.	$t^{-2/(1+\rho)} \times (\log t)^{-\rho/(1+\rho)}$ for both aggregates; the displayed coefficient is conditional on the face-transfer hypotheses of Theorem A.13 .		Integrable hidden pair face with power $2/(1+\rho) < 2$; the compact-face limit is proved, while uniform weighted-half-space transfer remains open.	No; the density is unbounded at equal-tail corners.
One-sided Gaussian, $-1 < \rho < 0$	Axes.	Undetermined. The nominal lower–lower hidden scale is not directly transferable because its face functional is nonintegrable.		The lower–lower hidden face is nonintegrable for the half-space increment, so an edge-renormalized term is expected. The leading expansion is open.	No globally.
Two-sided Gaussian, $\rho \neq 0$	Axes when $ \rho < 1$.	$t^{-2/(1+ \rho)} \times (\log t)^{- \rho /(1+ \rho)}$ for both aggregates; conditional on the face-transfer hypotheses of Theorem A.13 .		A dominant sign quadrant gives an integrable hidden pair face with power $2/(1+ \rho) < 2$; the coefficient is identified, but the noncompact boundary transfer remains open.	No.
Standard multivariate- t , $\nu > 2$	Two-sided scores have full-interior support; one-sided scores have mixed support with interior mass.	$t^{-1-2/\nu}$ for both aggregates and both sidedness choices; proved for fixed m .		Ordinary common-radial perturbation of the existing first-order measure; the $t^{-1-2/\nu}$ fixed- m expansion is proved for both aggregates in the standard model.	No; positive tail dependence forces an unbounded lower-tail density.
Standard multivariate- t , $1 < \nu \leq 2$	Two-sided scores have full-interior support; one-sided scores have mixed support with interior mass.	Two-sided reciprocal and both Half-Cauchy: $t^{-1-2/\nu}$. One-sided reciprocal: t^{-2} for $1 < \nu < 2$, with a tie at $\nu = 2$; proved for fixed m .		The same common-radial mechanism competes with an angular edge term. The two-sided reciprocal and both positive Half-Cauchy expansions retain the $t^{-1-2/\nu}$ term; the one-sided reciprocal expansion has an order t^{-2} angular term below $\nu = 2$, with a tie at $\nu = 2$.	No.
Standard multivariate- t , $0 < \nu \leq 1$	Two-sided scores have full-interior support; one-sided scores have mixed support with interior mass.	Undetermined; no aggregate second-order rate is proved.		Common-radial and angular subface terms meet or become nonintegrable; the second-order half-space problem is open.	No.

Continued on next page

Copula and parameters	pa- First-order support	Aggregate order rate	second- order rate	Mechanism and current status	Global bounded density
Clayton, $0 < \theta < 1$	Full-interior mass.	$t^{-1-\theta}$ for both aggregates; sharp and proved for fixed m .		Ordinary common-frailty perturbation with a proved $t^{-1-\theta}$ correction for both aggregates; it is larger than $t^{-2} \log t$.	No.
Clayton, $\theta \geq 1$	Full-interior mass.	Reciprocal: $t^{-1-\theta}$. Half-Cauchy: $t^{-1-\theta}$ for $1 \leq \theta < 2$ and t^{-3} for $\theta \geq 2$; proved for fixed m .		The reciprocal correction remains $t^{-1-\theta}$. The positive Half-Cauchy correction agrees below $\theta = 2$, ties its score-map term at $\theta = 2$, and has order t^{-3} above that boundary.	No.
Bivariate Clayton, $-1 \leq \theta < 0$	Axes; comparable lower extremes are eventually excluded.	Open for $-1 < \theta < 0$. At $\theta = -1$, reciprocal: t^{-2} ; Half-Cauchy: $O(t^{-3})$; proved.		Singular exclusion/edge geometry. The countermonotone endpoint is proved, but the interior negative-parameter family remains open.	No: unbounded for $-1 < \theta < 0$; the endpoint $\theta = -1$ is singular.
Gumbel, $\theta = 1$	Axes; independence.	$t^{-2} \log t$ for both aggregates; sharp and proved.		Critical hidden pair faces; the sharp independent logarithm is proved.	Yes.
Gumbel, $\theta > 1$	Axes.	Candidate $t^{-2^{1/\theta}}$ for both aggregates; the aggregate rate is open.		The joint-tail scaling suggests an integrable hidden pair-face correction of power $2^{1/\theta} \in (1, 2)$; only the rectangle obstruction to the target rate is proved here.	No; the density is unbounded at corners.
Joe, $\theta = 1$	Axes; independence.	$t^{-2} \log t$ for both aggregates; sharp and proved.		Critical hidden pair faces; the sharp independent logarithm is proved.	Yes.
Joe, $\theta > 1$ fixed	Axes, with $C(sx, sy) = O(s^2)$.	$t^{-2} \log t$ bivariate for both aggregates; sharp and proved. For $m > 2$ the same rate is conditional on the higher-face hypothesis.		In the bivariate case the sharp local-corner theorem applies with $\lambda = \theta$; in higher dimension its explicit higher-face condition still has to be verified.	No globally; unbounded at the upper corner.
Cuadras–Augé, $\theta = 0$	Axes; independence.	$t^{-2} \log t$ for both aggregates; sharp and proved.		Critical hidden pair faces; the sharp independent logarithm is proved.	Yes.
Cuadras–Augé, $0 < \theta < 1$	Axes.	Candidate $t^{-(2-\theta)}$ for both aggregates; the aggregate rate is open.		A singular hidden pair-face term of power $2 - \theta$ is expected; only the rectangle rate and singular-component diagnosis are proved.	No; the copula has a singular component.

Continued on next page

Copula and parameters	First-order support	diagonal ray	Aggregate order rate	second-order rate	Mechanism and current status	Global bounded density
Cuadras–Augé, $\theta = 1$	Full-interior	diagonal ray.	Reciprocal: no correction. Half-Cauchy: t^{-3} ; exact and proved.		No dependence perturbation remains after normalization: the reciprocal aggregate is exactly unit Pareto, while the positive Half-Cauchy aggregate has its exact univariate $O(t^{-3})$ score-map correction.	No; singular.
Marshall–Olkin with both non-trivial shock parameters	Axes in the bivariate lower-tail orientation used here; reflected or higher-dimensional shock models can charge proper faces or be mixed.		Aggregate rate undetermined. The equal-threshold rectangle scale is $t^{-\{2-\min(\alpha, \beta)\}}$ only.		A singular hidden pair-face power is expected axially; the weighted half-space expansion remains open.	No; it has a singular component.
Three-dimensional reflected Marshall–Olkin/max-linear pair-shock model	Pure proper faces.	pair	Reciprocal: $\sigma_2(\mathbf{w})t^{-2} \log t$; Half-Cauchy: $c^2 \sigma_2(\mathbf{w})t^{-2} \log t$, both with $O(t^{-2})$ remainder; proved.		Critical accumulation of two independent edge shocks near the charged proper faces; Theorem 6.3 gives the sharp expansion.	No; the model is max-linear and singular.
Comonotone copula	Full-interior	diagonal ray.	Reciprocal: no correction. Half-Cauchy: t^{-3} ; exact and proved.		Exact one-dimensional reduction: reciprocal aggregation is exactly unit Pareto and the positive Half-Cauchy score-map correction is proved.	No; singular.
Countermonotone bivariate copula	Axes; lower comparable extremes are excluded.	com-	Reciprocal: $2w(1-w)t^{-2}$. Half-Cauchy: $\frac{2}{\pi}\{2w(1-w)-1/3\}t^{-3} + O(t^{-5})$; the t^{-3} coefficient vanishes at $w = (1 \pm 1/\sqrt{3})/2$, leaving $O(t^{-5})$.		Singular exclusion geometry; the reciprocal correction is order t^{-2} , whereas the positive Half-Cauchy correction is generically order t^{-3} and can cancel at the displayed weights.	No; singular.

The ledger shows why support and mechanism must be recorded separately. Positive lower-tail dependence rules out global bounded density but does not rule out a quantitative half-space theorem: Clayton, standard multivariate- t , and perfect dependence all illustrate this distinction. Conversely, lower-tail independence identifies axis support but not the next rate: independence is critical, positive Gaussian and Gumbel models have larger hidden-face corrections, and singular Cuadras–Augé or Marshall–Olkin models require separate boundary analysis.

Rotations must be classified by which corner maps to small p -values. They preserve global boundedness, but can move a tail-dependent or corner-singular family’s singularity into the relevant lower corner and alter the Level-II conclusion.

B.1 Three reusable criteria

Let (U, V) have exact uniform margins and bivariate copula C . For fixed $x, y > 0$, inclusion–exclusion gives

$$\frac{1}{s}\mathbb{P}(U \leq sx \text{ or } V \leq sy) = x + y - \frac{C(sx, sy)}{s}. \quad (\text{B.1})$$

Consequently:

1. if $C(sx, sy)/s \rightarrow \Lambda_L(x, y)$, the lower-MDA limit exists and equals $x + y - \Lambda_L(x, y)$;
2. the rectangle form of the Level-II target is

$$\frac{C(sx, sy)}{s} - \Lambda_L(x, y) = O\{s \log(1/s)\}; \quad (\text{B.2})$$

3. in the axis-supported case $\Lambda_L = 0$, a single diagonal calculation

$$C(s, s) \neq O\{s^2 \log(1/s)\}$$

disproves the Level-II rectangle rate.

If C has a globally bounded density $c_C \leq M$, then

$$C(u, v) \leq Muv. \quad (\text{B.3})$$

[Theorem A.3](#) and [Corollary A.4](#) then give the full Level-II bound for both reciprocal/harmonic-mean and positive Half-Cauchy aggregation. Conversely, [Eq. \(B.3\)](#) shows that a copula satisfying $C(s, s) \gg s^2$ cannot have a globally bounded density. The same argument applies to any other corner rectangle: if

$$\mathbb{P}(U \leq s, V \geq 1 - s) \gg s^2 \quad \text{or} \quad \mathbb{P}(U \geq 1 - s, V \geq 1 - s) \gg s^2,$$

then global boundedness is impossible. A copula with a nonzero singular component is, of course, not absolutely continuous and cannot satisfy the density condition.

B.2 Product, FGM, and the smooth bounded-density families

Product copula For $C(u, v) = uv$,

$$C(sx, sy) = s^2xy, \quad c_C(u, v) = 1.$$

Thus $C(sx, sy)/s \rightarrow 0$, the lower-MDA measure is axis-supported, and the global density bound proves Level II. [Eqs. \(4.9\)](#) and [\(4.11\)](#) show that the logarithmic second term is generally genuinely present.

FGM copula For $-1 \leq \theta \leq 1$,

$$\begin{aligned} C_\theta(u, v) &= uv\{1 + \theta(1 - u)(1 - v)\}, \\ c_\theta(u, v) &= 1 + \theta(1 - 2u)(1 - 2v). \end{aligned}$$

Hence

$$C_\theta(sx, sy) = s^2xy\{1 + \theta + O(s)\}, \quad 0 \leq c_\theta(u, v) \leq 2.$$

The lower-MDA limit is axial and the bounded-density theorem proves the full Level-II rate.

AMH with $\theta < 1$ For

$$C_\theta(u, v) = \frac{uv}{1 - \theta(1 - u)(1 - v)}, \quad -1 \leq \theta < 1,$$

the denominator is bounded away from zero on $[0, 1]^2$ for every fixed $\theta < 1$. The copula and its two mixed derivatives are rational functions with that same nonvanishing denominator. Therefore the density is continuous and bounded on the compact square. Also,

$$C_\theta(sx, sy) = \frac{s^2xy}{1 - \theta + O(s)} = O(s^2). \quad (\text{B.4})$$

This proves the axial Level-I limit and the full Level-II result. The lower bound on the denominator tends to zero as $\theta \uparrow 1$, so this argument is not uniform at the endpoint.

Frank copula For fixed finite $\theta \neq 0$,

$$C_\theta(u, v) = -\frac{1}{\theta} \log \left[1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1} \right]. \quad (\text{B.5})$$

Taylor expansion at the lower corner yields

$$C_\theta(sx, sy) = \frac{\theta}{1 - e^{-\theta}} s^2xy + O(s^3). \quad (\text{B.6})$$

Its density is

$$c_\theta(u, v) = \frac{\theta(1 - e^{-\theta})e^{-\theta(u+v)}}{\{1 - e^{-\theta} - (1 - e^{-\theta u})(1 - e^{-\theta v})\}^2}. \quad (\text{B.7})$$

For fixed finite θ , the denominator in Eq. (B.7) is bounded away from zero in absolute value on the compact square. Thus the density is bounded. At $\theta = 0$ the continuous limit is the product copula. This proves all three Frank entries.

Plackett copula For $0 < \theta < \infty$, $\theta \neq 1$, write

$$D_\theta(u, v) = \{1 + (\theta - 1)(u + v)\}^2 - 4\theta(\theta - 1)uv,$$

$$C_\theta(u, v) = \frac{1 + (\theta - 1)(u + v) - \sqrt{D_\theta(u, v)}}{2(\theta - 1)}.$$

Direct differentiation gives

$$c_\theta(u, v) = \frac{\theta\{1 + (\theta - 1)(u + v - 2uv)\}}{D_\theta(u, v)^{3/2}}. \quad (\text{B.8})$$

For every fixed $\theta > 0$, D_θ is strictly positive on $[0, 1]^2$; continuity and compactness therefore give a finite density bound. Expanding the copula at $(0, 0)$ gives

$$C_\theta(sx, sy) = \theta s^2xy + O(s^3). \quad (\text{B.9})$$

The case $\theta = 1$ is the product copula. Hence the lower limit is axial and Level II follows from bounded density. The bound is parameterwise and need not remain uniform as $\theta \downarrow 0$ or $\theta \uparrow \infty$.

Finite Bernstein, checkerboard, and mixture constructions A fixed-degree Bernstein copula density is a finite sum of bounded beta polynomial basis functions. A finite checkerboard density is piecewise constant on finitely many cells. A finite mixture $c = \sum_{k=1}^K \pi_k c_k$ of bounded component densities satisfies

$$\|c\|_\infty \leq \sum_{k=1}^K \pi_k \|c_k\|_\infty < \infty.$$

Each construction therefore satisfies Level III, and hence Levels II and I, provided the degree, grid, number of components, and their bounds are fixed.

For the bivariate FGM, AMH with $\theta < 1$, Frank, and Plackett families, the displayed analytic formulas also give

$$c_\theta(u, v) = \lambda_\theta + O(u + v) \quad \text{as } (u, v) \rightarrow (0, 0),$$

with $\lambda_\theta = 1 + \theta$, $(1 - \theta)^{-1}$, $\theta/(1 - e^{-\theta})$, and θ , respectively. The Dini condition in [Theorem 5.5](#) therefore holds, and its higher-face condition is vacuous in two dimensions. This proves the sharp $t^{-2} \log t$ coefficient recorded in [Table 5](#) for both aggregates. In dimensions above two the same conclusion is conditional on the theorem's explicit higher-face bound; bounded pairwise margins alone do not verify it.

B.3 The AMH endpoint

At $\theta = 1$,

$$C_1(u, v) = \frac{uv}{u + v - uv}.$$

For $x + y > 0$,

$$\begin{aligned} \frac{C_1(sx, sy)}{s} &= \frac{xy}{x + y - sxy} \\ &= \frac{xy}{x + y} + s \frac{x^2 y^2}{(x + y)^2} + O(s^2). \end{aligned} \tag{B.10}$$

Thus

$$\Lambda_L(x, y) = \frac{xy}{x + y}, \quad \ell_L(x, y) = x + y - \frac{xy}{x + y},$$

and the rectangle error is $O(s)$, which is compatible with [Eq. \(B.2\)](#). This rectangle calculation alone proves only compatibility; the bounded-density theorem is unavailable. Indeed, direct differentiation gives

$$c_1(u, v) = \frac{2uv}{(u + v - uv)^3}, \tag{B.11}$$

so $c_1(s, s) \sim (4s)^{-1}$ and Level III fails.

The endpoint is not a separate unresolved model. Algebraically,

$$C_{\text{AMH},1}(u, v) = \frac{uv}{u + v - uv} = (u^{-1} + v^{-1} - 1)^{-1} = C_{\text{Clayton},1}(u, v). \tag{B.12}$$

Consequently, with $v = 1 - w$ and $0 < w < 1$, [Theorems 8.2](#) and [8.3](#) give

$$\mathbb{P}(wY_1 + vY_2 > t) = \frac{1}{t} + wv t^{-2} + o(t^{-2}), \tag{B.13}$$

$$\mathbb{P}(wH_1 + vH_2 > t) = \frac{c}{t} + c^2 wv t^{-2} + o(t^{-2}). \tag{B.14}$$

Thus both sharp coefficients are already part of the proved positive Clayton common-frailty theorem.

B.4 Clayton copulas

Positive parameter For $\theta > 0$,

$$C_\theta(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}.$$

Let $A = x^{-\theta} + y^{-\theta}$. Then

$$\begin{aligned} \frac{C_\theta(sx, sy)}{s} &= (A - s^\theta)^{-1/\theta} \\ &= A^{-1/\theta} + \frac{1}{\theta} A^{-1/\theta-1} s^\theta + O(s^{2\theta}). \end{aligned} \quad (\text{B.15})$$

Hence the lower-MDA limit exists with

$$\Lambda_L(x, y) = (x^{-\theta} + y^{-\theta})^{-1/\theta}.$$

When $0 < \theta < 1$, the error s^θ is larger than $s \log(1/s)$, so the Level-II rectangle diagnostic fails. When $\theta \geq 1$, the error is $O\{s \log(1/s)\}$, so the rectangle diagnostic is compatible. The separate half-space argument is supplied by [Theorems 8.2](#) and [8.3](#). For R_w the correction has order $t^{-1-\theta}$ for every $\theta > 0$. For T_w it has the same order below $\theta = 2$, ties the intrinsic score-map term at $\theta = 2$, and has order t^{-3} above that boundary. Thus Level II holds for both aggregates whenever $\theta \geq 1$.

For every $\theta > 0$, $C_\theta(s, s) \sim 2^{-1/\theta} s$. A globally bounded density would instead force $C_\theta(s, s) = O(s^2)$ by [Eq. \(B.3\)](#). Thus Level III fails for every positive Clayton parameter.

Negative parameter Write $\theta = -\alpha$ with $0 < \alpha \leq 1$. The bivariate copula is

$$C_{-\alpha}(u, v) = (u^\alpha + v^\alpha - 1)_+^{1/\alpha}. \quad (\text{B.16})$$

For fixed $x, y > 0$, $s^\alpha(x^\alpha + y^\alpha) < 1$ eventually, so

$$C_{-\alpha}(sx, sy) = 0$$

for all sufficiently small s . The lower-MDA measure is therefore axial, and the rectangle diagnostic is exact near the lower corner. This does not by itself prove the weighted half-space rate because the support boundary can be encountered at unequal scales.

For $0 < \alpha < 1$, the density on $u^\alpha + v^\alpha > 1$ is

$$c_{-\alpha}(u, v) = (1 - \alpha)(uv)^{\alpha-1}(u^\alpha + v^\alpha - 1)^{1/\alpha-2}. \quad (\text{B.17})$$

Take

$$v = (1 - u^\alpha/2)^{1/\alpha}.$$

Then $v \rightarrow 1$, the last parenthesis in [Eq. \(B.17\)](#) equals $u^\alpha/2$, and

$$c_{-\alpha}(u, v) \asymp u^{-\alpha} \rightarrow \infty.$$

Thus Level III fails throughout $-1 < \theta < 0$. At $\theta = -1$ the copula is the singular countermonotone copula.

B.5 Gumbel and Joe copulas

Gumbel For $\theta \geq 1$,

$$C_\theta(u, v) = \exp[-\{(-\log u)^\theta + (-\log v)^\theta\}^{1/\theta}].$$

At $\theta = 1$ this is the product copula. Suppose $\theta > 1$ and let $L = \log(1/s)$ and $q = 2^{1/\theta}$. For fixed $x, y > 0$,

$$\{(L - \log x)^\theta + (L - \log y)^\theta\}^{1/\theta} = qL - \frac{q}{2}(\log x + \log y) + O(L^{-1}).$$

Consequently,

$$C_\theta(sx, sy) = s^q(xy)^{q/2}\{1 + O(L^{-1})\} = o(s), \quad (\text{B.18})$$

so the lower-MDA limit is axial. On the diagonal the formula is exact:

$$C_\theta(s, s) = s^{2^{1/\theta}}, \quad 1 < 2^{1/\theta} < 2. \quad (\text{B.19})$$

Hence $C_\theta(s, s)/s$ converges too slowly for [Eq. \(B.2\)](#), proving Level-II failure. [Eq. \(B.19\)](#) also contradicts the $O(s^2)$ bound implied by a globally bounded density, so Level III fails.

The sharper expansion in [Eq. \(B.18\)](#) has the product-power hidden survival $(xy)^{-a}$ after reciprocal scoring, with $a = q/2 \in (1/2, 1)$. The corresponding Möbius half-space increment is therefore integrable by [Proposition 5.3](#). This supports the mechanism prediction in [Table 5](#); it does not by itself verify the uniform truncation needed for a full weighted-sum expansion. Because $q < 2$, any validated reciprocal correction at this scale would transfer through [Proposition 2.1](#) to T_w with coefficient scaled by c^q . The missing truncation argument is therefore open for both aggregates, not only for the reciprocal scores.

Joe For $\theta \geq 1$,

$$C_\theta(u, v) = 1 - \{(1-u)^\theta + (1-v)^\theta - (1-u)^\theta(1-v)^\theta\}^{1/\theta}. \quad (\text{B.20})$$

At $\theta = 1$ this is the product copula. For fixed $x, y > 0$ and $\theta > 1$, Taylor expansion at the lower corner gives

$$C_\theta(sx, sy) = \theta s^2 xy + O(s^3). \quad (\text{B.21})$$

Thus the lower-MDA limit is axial. To verify the lower-strip condition, put

$$a = (1-u)^\theta, \quad b = (1-v)^\theta, \quad D = a + b - ab.$$

Differentiation of [Eq. \(B.20\)](#) gives

$$c_\theta(u, v) = (1-u)^{\theta-1}(1-v)^{\theta-1}D^{1/\theta-2} \times \{\theta D + (\theta-1)(1-a)(1-b)\}. \quad (\text{B.22})$$

If $u \leq u_0 < 1$, then $a \geq (1-u_0)^\theta > 0$ and $D \geq a$. Every factor in [Eq. \(B.22\)](#) is therefore uniformly bounded for $u \leq u_0$ and $0 < v < 1$; symmetry handles the other lower strip. [Theorem A.3](#) with the strip observation [Eq. \(A.19\)](#) proves the full Level-II rate. Moreover, expansion of [Eq. \(B.22\)](#) at the lower corner gives $c_\theta(u, v) = \theta + O(u+v)$. Hence the bivariate Dini condition in [Theorem 5.5](#) holds with $\lambda = \theta$, proving the sharp logarithmic coefficient for both aggregates. For $m > 2$, the same coefficient remains conditional on the theorem's higher-face hypothesis.

Global boundedness nevertheless fails. From [Eq. \(B.20\)](#),

$$\begin{aligned} \mathbb{P}(U > 1-s, V > 1-s) &= 1 - 2(1-s) + C_\theta(1-s, 1-s) \\ &= s\{2 - (2-s^\theta)^{1/\theta}\} \\ &\sim \{2 - 2^{1/\theta}\}s \asymp s. \end{aligned} \quad (\text{B.23})$$

This is incompatible with the $O(s^2)$ upper-corner probability implied by a globally bounded density.

B.6 Cuadras–Augé, Marshall–Olkin, and the Fréchet endpoints

Cuadras–Augé For

$$C_\theta(u, v) = \{\min(u, v)\}^\theta (uv)^{1-\theta}, \quad 0 \leq \theta \leq 1,$$

assume without loss that $x \leq y$. Then

$$C_\theta(sx, sy) = s^{2-\theta} xy^{1-\theta}. \quad (\text{B.24})$$

At $\theta = 0$ this is the product copula. For $0 < \theta < 1$, $C_\theta(sx, sy) = o(s)$, so the lower-MDA measure is axial, but

$$\frac{C_\theta(sx, sy)}{s} = s^{1-\theta} xy^{1-\theta}$$

is too large for the Level-II rectangle rate. The copula has a nonzero singular component for every $\theta > 0$, so Level III fails. At $\theta = 1$, $C_1(u, v) = \min(u, v)$ is the comonotone copula treated below.

Marshall–Olkin For $0 \leq \alpha, \beta \leq 1$,

$$C_{\alpha, \beta}(u, v) = \min\{u^{1-\alpha}v, uv^{1-\beta}\}. \quad (\text{B.25})$$

For fixed $x, y > 0$,

$$C_{\alpha, \beta}(sx, sy) = \min\{s^{2-\alpha}x^{1-\alpha}y, s^{2-\beta}xy^{1-\beta}\}. \quad (\text{B.26})$$

If $\alpha, \beta > 0$ and $(\alpha, \beta) \neq (1, 1)$, this has diagonal order

$$C_{\alpha, \beta}(s, s) = s^{2-\min(\alpha, \beta)}, \quad 1 < 2 - \min(\alpha, \beta) < 2. \quad (\text{B.27})$$

Therefore the lower-MDA limit is axial but the Level-II rectangle rate fails. The copula has a singular component on $u^\alpha = v^\beta$, so Level III fails. The endpoint $\alpha = \beta = 1$ is comonotone. If either shock parameter is zero, the diagonal exponent is two and the failure argument in Eq. (B.27) no longer applies; this is why the table states both parameters to be nontrivial.

Comonotone copula If $U_1 = \dots = U_m = U$, then the copula is comonotone, the lower tail measure is diagonal, and normalized weights give

$$R_w = U^{-1}, \quad \mathbb{P}(R_w > t) = t^{-1}, \quad t \geq 1,$$

whereas

$$T_w = \cot(\pi U/2).$$

Hence

$$\begin{aligned} \mathbb{P}(T_w > t) &= 1 - \frac{2}{\pi} \arctan t \\ &= \frac{2}{\pi t} - \frac{2}{3\pi t^3} + O(t^{-5}). \end{aligned} \quad (\text{B.28})$$

Thus reciprocal aggregation is exactly Pareto, while positive Half-Cauchy aggregation has the sharper $O(t^{-3})$ remainder. The copula is singular, so Level III fails. The same calculation proves the Cuadras–Augé endpoint entry.

Countermonotone copula Let $(U_1, U_2) = (U, 1 - U)$ and let $0 \leq w \leq 1$. Put

$$H = \cot(\pi U/2).$$

Then H is standard Half-Cauchy and

$$\cot\{\pi(1 - U)/2\} = H^{-1}, \quad T_w = wH + (1 - w)H^{-1}.$$

The reciprocal calculation is already given in Eq. (6.9); it has an order- t^{-2} correction. The positive Half-Cauchy correction is different: For $0 < w < 1$ and $t > 2\sqrt{w(1 - w)}$, define

$$h_{\pm}(t) = \frac{t \pm \sqrt{t^2 - 4w(1 - w)}}{2w}. \quad (\text{B.29})$$

The event $\{T_w > t\}$ is the disjoint union $\{H < h_-(t)\} \cup \{H > h_+(t)\}$. Since $F_H(h) = 2 \arctan(h)/\pi$,

$$\mathbb{P}(T_w > t) = \frac{2}{\pi} \left\{ \arctan h_-(t) + \arctan \frac{1}{h_+(t)} \right\}. \quad (\text{B.30})$$

Expansion of Eq. (B.29) gives

$$h_-(t) = \frac{1 - w}{t} + \frac{w(1 - w)^2}{t^3} + O(t^{-5}), \quad h_+(t)^{-1} = \frac{w}{t} + \frac{w^2(1 - w)}{t^3} + O(t^{-5}).$$

Using $\arctan x = x - x^3/3 + O(x^5)$ therefore gives

$$\mathbb{P}(T_w > t) = \frac{2}{\pi t} + \frac{2}{\pi} \left\{ 2w(1 - w) - \frac{1}{3} \right\} t^{-3} + O(t^{-5}). \quad (\text{B.31})$$

The cases $w = 0$ and $w = 1$ reduce directly to a standard Half-Cauchy variable. Also,

$$C_W(sx, sy) = \max\{s(x + y) - 1, 0\} = 0$$

for all sufficiently small s , proving the exact axial lower-MDA limit. The countermonotone copula is singular, so Level III fails.

B.7 Gaussian and standard multivariate- t copulas

One-sided Gaussian copula Theorem 7.1 already proves Level I for every fixed $|\rho| < 1$. In the bivariate case, the sharper equal-threshold asymptotic is

$$C_\rho(s, s) \asymp s^{2/(1+\rho)} \{\log(1/s)\}^{-\rho/(1+\rho)}. \quad (\text{B.32})$$

For $0 < \rho < 1$, the power $2/(1 + \rho)$ lies strictly between one and two, so the Level-II rectangle diagnostic fails. The same display contradicts the $O(s^2)$ implication of global bounded density.

For $-1 < \rho < 0$, the power in Eq. (B.32) exceeds two. Thus the lower-lower rectangle error is $o\{s \log(1/s)\}$ and is compatible with Level II. It does not complete the half-space proof, because the Gaussian copula density is singular at opposite corners. Indeed, writing $z_u = \Phi^{-1}(u)$, the density is

$$c_\rho(u, v) = \frac{1}{\sqrt{1 - \rho^2}} \exp \left[\frac{2\rho z_u z_v - \rho^2(z_u^2 + z_v^2)}{2(1 - \rho^2)} \right]. \quad (\text{B.33})$$

For $\rho > 0$, take $u = v \downarrow 0$; for $\rho < 0$, take $u \downarrow 0$ and $v \uparrow 1$ with $z_v = -z_u$. The exponent tends to $+\infty$ in each case, proving that the global density is unbounded. At $\rho = 0$, Eq. (B.33) equals one and the copula is the product copula.

For $0 < \rho < 1$, the aggregate rate in Table 5 is not deduced from the equal-threshold rectangle alone. The required unequal-tail approximation and integrable face functional identify the candidate coefficient. Truncation control and the higher-face remainder are explicit assumptions of Theorem A.13, not proved consequences of the rectangle limit.

Two-sided Gaussian copula For $p_i = 2\{1 - \Phi(|Z_i|)\}$, the joint small- p event is the union of four sign quadrants. A quadrant whose signed Gaussian correlation is $|\rho|$ gives a lower bound of order

$$s^{2/(1+|\rho|)} \{\log(1/s)\}^{-|\rho|/(1+|\rho|)},$$

while the four-quadrant union bound gives the same power as an upper bound. Thus Level I remains axial for $|\rho| < 1$, but the Level-II rectangle rate fails whenever $\rho \neq 0$. The induced two-sided p -value copula cannot have a globally bounded density, since boundedness would force every lower-corner square probability to be $O(s^2)$. The rate and coefficient recorded in Table 5 are conditional on the two-sided face-transfer assumptions of Theorem A.13; they do not follow from this rectangle diagnostic alone.

Standard multivariate- t copula Theorem A.15 proves index-one MRV of the reciprocal and Half-Cauchy score vectors. Its spectral measure generally has interior mass. In the bivariate case,

$$\frac{C_{t,\rho,\nu}(s,s)}{s} \longrightarrow \lambda_t = 2T_{\nu+1} \left(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}} \right) > 0.$$

If the copula density were globally bounded, the numerator would be $O(s^2)$, a contradiction. Hence Level III fails. Ordinary MRV supplies no numerical second-order rate. The model-specific calculation in Theorem 8.1 supplies the missing mapping and half-space argument for $\nu > 1$. For $\nu > 2$, both R_w and T_w have a radial correction of order $t^{-1-2/\nu}$. For one-sided R_w , an angular order- t^{-2} term ties the radial term at $\nu = 2$ and dominates it for $1 < \nu < 2$; two-sided R_w and either orientation of T_w retain the radial order. The cases $0 < \nu \leq 1$ remain open because of angular criticality.

B.8 Rotations and completion of the table audit

If an absolutely continuous copula is reflected in one or both coordinates, its rotated density is obtained from c_C by replacing u with $1 - u$ and/or v with $1 - v$. Thus a global density bound is preserved exactly under rotations. Lower-tail behavior is not preserved: the reflection permutes the lower–lower, lower–upper, upper–lower, and upper–upper corners. Consequently, the lower-MDA and Level-II classification of a rotated tail-dependent or corner-singular family must be recomputed at the corner mapped to small p -values. This proves the rotation statement following Table 5 and completes the classification audit.

C Literature boundary and proof-status ledger

This appendix consolidates the extended literature review and the claim-by-claim novelty audit. It separates classical foundations and the closest existing results from the statements proved here and from the questions that remain open. Here *second-order calibration* means the leading nonzero difference between the dependent tail and the exact independence tail after their common t^{-1} term. This usage is broader than classical second-order MRV: the correction may instead come from a hidden or critical face, a common radial or angular expansion, or the Half-Cauchy score map.

C.1 What was already known

The following inputs belong to the literature and should not be claimed as new.

1. Reciprocal aggregation underlies the harmonic-mean p -value of Wilson [41]. Positive Half-Cauchy aggregation belongs to the heavily right framework of Liu, Meng and Pillai [33], building on the signed

Cauchy combination test of Liu and Xie [34] and its subsequent dependence analysis [7]. The signed and positive transformations are not the same problem: the nonnegative statistics considered here are governed by the lower tail of the p -value copula, whereas signed scores can involve cancellation and opposite-corner conditions.

2. Homogeneous mapping of MRV vectors, spectral representations, and weighted-sum first-order tails are classical; see, for example, Hult and Lindskog [6] and Basrak, Davis and Mikosch [1]. Therefore the fixed-dimensional statement that index-one MRV implies a universal normalized projection constant is a foundation, not the main novelty. For heavy-tailed combination tests specifically, first-order fixed- m validity under quasi-asymptotic independence is given by Gui, Jiang and Wang [27]; the MRV-copula benchmark under asymptotic dependence is given by Gui, Mao, Wang and Wang [28]; and the angular-measure formulation and universal Pareto/harmonic-mean calibration result are given by Chakraborty, Guo, Shedden and Stoev [22]. The one-big-jump argument of Liu, Meng and Pillai [33] is a sufficient route through pairwise tail control, but it is not necessary for the index-one projection identity: interior-supported models can retain the same first-order constant.
3. Transfer from multivariate second-order regular variation to risk aggregation is studied by Das and Kratz [11]. A generic assertion that second-order MRV gives a second-order sum expansion is not new. Second-order convolution theory and risk-concentration expansions also predate this paper [26, 21, 25]. Yang and Zhang [42] explicitly obtain the bivariate unit-Pareto $(\log t)/t^2$ term and the corresponding edge boundary. Thus the logarithm itself is not new. The issue specific to the present problem is its arbitrary fixed-dimensional, unequal-weight, exactly standardized harmonic-mean/positive-Half-Cauchy calibration coefficient with explicit higher-face control.
4. Hidden regular variation, regular variation on nested cones, residual tail orders, and higher-order tail densities are established frameworks [30, 31, 37, 38, 29, 32]. Möbius inversion of face and excess measures is likewise classical [20]. The possible contribution here is the particular weighted-half-space increment and its second-order calibration use, not the underlying cone or inclusion–exclusion machinery.
5. Gaussian-copula risks with regularly varying margins and their hidden tail geometry have substantial prior theory, including Kortschak [12] and Das and Fasen-Hartmann [4]. The overlap is substantial at the level of bivariate joint-tail powers and Gaussian cone optimization. The present Gaussian calculation should be described only as a conditional reciprocal weighted-half-space face integral, its transfer to positive Half-Cauchy aggregation under the same boundary conditions, its formal arbitrary- m sum over dominant pairs, the two-sided quadrant constant, and the critical transition to the independent logarithm. A theorem-by-theorem comparison with Kortschak remains necessary before a novelty claim is finalized.
6. Frailty representations and lower-tail properties of multivariate Archimedean copulas are standard; see McNeil and Nešlehová [13] and Charpentier and Segers [14]. Tail aggregation under Archimedean dependence also predates this paper; see Barbe, Fougères and Genest [15] and Embrechts, Nešlehová and Wüthrich [16], as well as the explicit second-order work of Tong, Wu and Xu [39] and Coqueret [24]. Finite-level harmonic-mean inequalities for some Clayton copulas are given by Chen, Wang, Wang and Zhu [23]. The Clayton claim here is therefore the exact R_w and T_w coefficients and the score-map phase diagram, not the discovery of its tail dependence, frailty representation, or finite-level validity bounds.
7. MRV and second-order radial behavior of elliptical and random-scale models are known [6, 2, 36, 10]. The standard- t contribution here is the exact uniform- p score expansion, explicit weighted angular

coefficient, and distinct one- versus two-sided reciprocal and Half-Cauchy corrections.

Table 6: Closest-result comparison delimiting the paper’s contribution. The last column states a scope boundary, not an unconditional priority claim.

Reference	Established overlap	Scope of the present contribution
Gui, Jiang and Wang [27]	Fixed- m first-order validity and Bonferroni equivalence under quasi-asymptotic independence.	Dependence-specific second-order size distortion under axis, interior, proper-face, and mixed MRV geometry.
Gui, Mao, Wang and Wang [28]	MRV-copula formulation; first-order validity and power under asymptotic dependence.	Rates and coefficients after the first-order MRV benchmark.
Chakraborty, Guo, Shedden and Stoev [22]	Angular-measure formula and universal first-order Pareto/harmonic-mean calibration within a broad homogeneous class.	Exact-independence-relative positive Half-Cauchy and harmonic-mean error, including its sign and critical-value effect.
Das and Kratz [11]	General second-order MRV transfer and risk-concentration theory.	Exactly standardized p -scores and non-compact weighted-half-space functionals whose face limits can fail to be integrable.
Kortschak [12]	Second-order sums for tail-independent risks, including Gaussian and some Archimedean copulas.	Only the exact positive Half-Cauchy and harmonic-mean normalization, arbitrary- m weighted coefficient, and model-specific calibration phases are potentially new; direct theorem comparison remains required.
Yang and Zhang [42]	Bivariate copula-sum expansions, the unit-index independence logarithm, and an integrable-versus-truncated-moment edge boundary.	The local-corner theorem extends the calculation to arbitrary fixed m , unequal weights, pair-specific kernels, and explicit higher-face control.
Tong, Wu and Xu [39]; Coqueret [24]	Second-order Archimedean risk concentration and Bernstein-copula aggregation.	The reciprocal- p /HC coefficients and Clayton score-map transition are specific contributions, conditional on formula-level comparison.
Balkema and Embrechts [20]	Geometric face/excess measures and Möbius inversion.	The induced weighted-half-space increment and its calibration use, not generic face-lattice calculus.
Marshall Olkin [17]; Wang and Stoev [40]	Shock models and discrete max-linear spectral support on selected faces.	The explicit three-edge weighted-sum coefficient and the comparison of positive Half-Cauchy with harmonic-mean calibration, pending a specialized aggregation search.
Chen, Wang, Wang and Zhu [23]	Finite-level subuniformity and threshold bounds, including some Clayton copulas.	Small-level asymptotic coefficients and phase transitions rather than finite-level validity inequalities.

This audit reflects sources located through August 23, 2026. It supports the scope statements above, but it does not replace a final formula-level comparison with the specialized Gaussian, Archimedean, and Marshall–Olkin aggregation literature before submission.

C.2 What is proved in this paper

The formal results now establish the following package.

1. Fixed-dimensional first-order validity of both reciprocal/harmonic-mean and positive Half-Cauchy aggregation under general index-one MRV/lower MDA, including exponent measures with interior mass. This is retained as a first-order benchmark, not a novelty claim.
2. An independence-relative tail-distortion definition and a rigorous conversion of a regularly varying tail difference into size and critical-value expansions.
3. A quantitative MRV projection result, product-shell and density criteria, an exact face-lattice decomposition, and an integrable hidden-face transfer theorem.
4. A support-first exponent-measure taxonomy, a proof that exact Pareto normalization rules out nonzero pure-axis second-order measures, and an explicit product-power trichotomy for integrable, critical, and nonintegrable pair-face functionals. The interior category also formally shows that every subset joint extreme is already first-order and that a score map homogeneous to the raw radial index converts the common radial variable to index one.
5. A rigorous common-heavy-factor aggregation theorem with explicit L^1 inversion assumptions, together with reciprocal-edge and Half-Cauchy phase diagrams that separate radial, angular, and score-map corrections.
6. The sharp independent coefficients multiplying $(\log t)/t^2$: $2 \sum_{i < j} w_i w_j$ for R_w and $2c^2 \sum_{i < j} w_i w_j$ for T_w .
7. The local-corner theorem in [Theorem 5.5](#): arbitrary fixed dimension and unequal weights, pair-specific finite corner limits, explicit control of every higher face, and the resulting R_w and T_w calibration coefficients and sign.
8. For Gaussian copulas, the compact hidden pair limits and their explicit Möbius coefficients, together with a conditional fixed- m half-space theorem that isolates the still-unproved stratified boundary estimates. No unconditional arbitrary-dimensional Gaussian expansion is claimed.
9. For the standard multivariate- t copula with $\nu > 1$, proved reciprocal/harmonic-mean and positive Half-Cauchy second-order expansions with computable Gaussian angular expectations, including their necessary one-sided distinction when $1 < \nu \leq 2$.
10. For every positive Clayton parameter and arbitrary fixed m , an exact reciprocal-sum correction with a nonnegative coefficient, strictly positive when at least two weights are active and zero for one active coordinate, together with the positive Half-Cauchy phase transition at $\theta = 2$.
11. The proper-face pair-shock theorem in [Theorem 6.3](#): a pure edge-supported exponent measure, an explicit $(\log t)/t^2$ coefficient, and its independence-relative calibration consequence. This is a canonical model theorem, not a general proper-face theorem.
12. Uniform projection and radial-kernel theorems, plus unrestricted growing-dimensional first-order validity for fixed-parameter positive Clayton and standard multivariate- t common-factor arrays.

C.3 What remains open

The following items are not proved and are deliberately not hidden inside the theorem statements.

1. A full nested critical-face theorem covering simultaneous critical and nonintegrable subfaces. The local-corner theorem controls the finite pair-kernel boundary under explicit higher-face bounds, but it does not supply the desired recursive theory for singular or nonintegrable nested faces.

2. A general proper-face or mixed-support second-order theorem beyond the explicit three-dimensional pair-shock calculation.
3. One-sided Gaussian second terms when all active correlations are nonpositive with at least one strictly negative, the stratified boundary estimate needed for the conditional positive-correlation theorem, and a uniform Gaussian theorem across $\rho \log t = O(1)$. The all-zero case is independence and is already proved.
4. The critical and subcritical standard- t cases $\nu \leq 1$.
5. Uniform second-order remainders for growing Clayton and t arrays. The coefficient bounds are dimension-free, but coefficient control is not remainder control.
6. A growing-dimensional Gaussian theorem with explicit accumulation of every active face. The polynomial rate following [Open problem A.25](#) is only a diagnostic.
7. Formula-level, theorem-by-theorem comparisons with the closest aggregation results for Gaussian, Archimedean, and Marshall–Olkin models before any unconditional priority claim is made.

D Proofs of the main results

The proofs are collected in the order in which their statements appear in the main text.

D.1 Heavily right scores and exponent-measure geometry

Proof of [Proposition 2.1](#). The expansion at zero is

$$\cot\left(\frac{\pi u}{2}\right) = \frac{2}{\pi u} - \frac{\pi u}{6} + O(u^3).$$

Thus $d(u) \rightarrow 0$ as $u \downarrow 0$. If $x = \pi u/2$, then

$$d'(u) = \frac{\pi}{2} \{\csc^2 x - x^{-2}\} > 0,$$

because $\sin x < x$ for $0 < x \leq \pi/2$. Since the continuous extension satisfies $d(0) = 0$ and $d(1) = c$, one has $0 \leq d \leq c$. Applying the scalar bound coordinatewise proves [Eq. \(2.2\)](#). Moreover,

$$\frac{\|\mathbf{H} - c\mathbf{Y}\|_\infty}{t} \leq \frac{c}{t} \rightarrow 0$$

deterministically. Bounded perturbations do not alter vague tail convergence on sets bounded away from the origin. This proves the equivalence and the scaling relation for μ_H . Its marginal masses follow directly from the displayed Half-Cauchy survival expansion in [Section 2.1](#). $\square$

Proof of [Theorem 2.3](#). For $\mathbf{Y}$, exact Pareto margins give $c_i = 1$; for $\mathbf{H}$, [Proposition 2.1](#) gives $c_i = 2/\pi$. Substitution into [Eq. \(2.12\)](#), together with $\sum_i w_i = 1$, gives the two constants. The radial measure has no atoms, so $\mu(\partial A_w) = 0$ and [Definition 2.2](#) applies to the half-space. Under independence, both score vectors are MRV with axial spectral measures, and the same calculation gives the two reference tails. $\square$

D.2 Lower max-domains for p-value copulas

Proof of Theorem 3.2. For $s > 0$ small, write

$$L_s(\mathbf{x}) = s^{-1} \mathbb{P} \left(\bigcup_i \{U_i \leq sx_i\} \right).$$

Exact uniformity and a symmetric-difference bound give

$$|L_s(\mathbf{x}) - L_s(\mathbf{y})| \leq \sum_i |x_i - y_i|.$$

Hence the pointwise limit ℓ_L is Lipschitz continuous. It is also homogeneous of order one. By Eq. (A.6), $\exp\{-\ell_L(\mathbf{x})\}$ is the continuous pointwise limit of the joint survival functions of normalized componentwise minima. After taking reciprocals,

$$G(\mathbf{y}) = \exp\{-\ell_L(1/y_1, \dots, 1/y_m)\}$$

is a proper, nondegenerate unit-Fréchet distribution; the exact marginal bounds give properness, and homogeneity of ℓ_L gives max-stability. The standard exponent-measure representation of a max-stable law therefore supplies a unique Radon measure μ_Y on $\mathbb{E}_m$; see, for example, Resnick [38]. For $\mathbf{x} \geq 0$, define

$$B_{\mathbf{x}} = \{\mathbf{y} \in \mathbb{E}_m : y_i > x_i^{-1} \text{ for at least one } i\},$$

where a coordinate with $x_i = 0$ contributes no event. The representation satisfies $\mu_Y(B_{\mathbf{x}}) = \ell_L(\mathbf{x})$, and

$$\begin{aligned} t \mathbb{P}(\mathbf{Y}/t \in B_{\mathbf{x}}) &= t \mathbb{P}\left(U_i < \frac{x_i}{t} \text{ for at least one } i\right) \\ &\longrightarrow \ell_L(\mathbf{x}). \end{aligned}$$

The sets $B_{\mathbf{x}}$ form a convergence-determining class for boundedly finite measures on $\mathbb{E}_m$ and are μ_Y -continuity sets; the latter also follows directly from the diffuse radial component of an exponent measure. Their convergence therefore gives the vague convergence in Definition 2.2. Proposition 2.1 transfers the convergence to $\mathbf{H}$. $\square$

Proof of Proposition 3.3. The union bound supplies the upper bound in Eq. (3.10). Bonferroni's lower bound differs from the sum of marginal probabilities by at most the finite sum of pairwise intersections. Exact uniformity and Eq. (3.9) give Eq. (3.10). Invoking Theorem 3.2 now proves MRV. Finally, Eq. (2.7) and the representation of ℓ_L uniquely identify the spectral measure: the identity $\ell_L(\mathbf{x}) = \sum_i x_i$ is precisely the unit mass constraint carried separately by the coordinate axes. Proposition 2.1 transfers the same support geometry to $\mathbf{H}$. $\square$

Proof of Corollary 3.4. Apply Theorems 2.3 and 3.2. $\square$

D.3 Second-order calibration and quantitative tail control

Proof of Theorem 4.1. For $q_0 = c_0/\alpha$, set

$$x_j = q_0 \left[1 + \frac{d_{j,w}}{c_0} q_0^{-\kappa} L(q_0) \right].$$

Since $x_j/q_0 \rightarrow 1$, the uniform convergence theorem for slowly varying functions gives $L(x_j)/L(q_0) \rightarrow 1$. Expanding c_0/x_j and the second term in Eq. (4.4) shows that $\bar{F}_{j,w}(x_j) = \alpha + o\{\alpha q_0^{-\kappa} L(q_0)\}$. For a fixed $\varepsilon > 0$, instead set

$$x_{j,\pm} = q_0 \left[1 + \left\{ \frac{d_{j,w}}{c_0} \pm \varepsilon \right\} q_0^{-\kappa} L(q_0) \right].$$

The same expansion gives

$$\bar{F}_{j,w}(x_{j,+}) < \alpha < \bar{F}_{j,w}(x_{j,-})$$

for all sufficiently small α . Monotonicity brackets the quantile between $x_{j,-}$ and $x_{j,+}$; letting $\varepsilon \downarrow 0$ proves Eq. (4.5), including when $d_{j,w} = 0$.

Subtracting the two instances of Eq. (4.4) and evaluating at $q_{\Pi,w}(\alpha) \sim q_0(\alpha)$ gives Eq. (4.6). Division by α gives Eq. (4.7), and subtraction of the two quantile expansions gives Eq. (4.8). $\square$

Proof of Theorem 4.2. For two coordinates, conditioning on Y_1 reduces the cooperative part to

$$\frac{w_1}{t - w_2} + w_1 w_2 \int_{w_1}^{t-w_2} \frac{dz}{z^2(t-z)}.$$

Partial fractions give Eq. (4.10). The logarithm is not a marginal remainder: both Pareto margins are exact. After scaling by t , the hidden product density is proportional to $t^{-2}x^{-2}y^{-2}$. Near, say, $y = 0$, the cooperative band cut out by the weighted-sum boundary has x -width proportional to y . Its truncated face integral is therefore

$$t^{-2} \int_{1/t}^1 y y^{-2} dy = t^{-2} \log t.$$

The other axis supplies the second copy. Equivalently, expanding Eq. (4.10) yields $2w_1 w_2 (\log t)/t^2$. For general fixed m , use the exact Möbius decomposition in Lemma 5.1. The singleton increments sum to t^{-1} . Applying the bivariate identity to each pair increment gives the displayed logarithmic coefficient, while Lemma 5.4 bounds every fixed face of dimension at least three by $O(t^{-2})$. Summing the finitely many face increments proves Eq. (4.9).

Finally, Proposition 2.1 gives a fixed K such that

$$\mathbb{P}(cR_w > t + K) \leq \mathbb{P}(T_w > t) \leq \mathbb{P}(cR_w > t - K).$$

A fixed threshold shift changes the leading c/t term by $O(t^{-2})$ and leaves the $(\log t)/t^2$ coefficient unchanged, proving Eq. (4.11). $\square$

Proof of Proposition A.2. If $m = 1$, the conclusion is the exact one-dimensional marginal expansion, so suppose $m \geq 2$. Taking $(a, b) = (t, \infty)$ by monotone approximation in Eq. (A.15) gives

$$\mathbb{P}(Z_i > t, Z_j > t) = O(t^{-2}).$$

Thus fixed-dimensional Bonferroni bounds make the union of individual big-jump events $c_X/t + O(t^{-2})$.

On the event

$$\left\{ \sum_i Z_i > t, \max_i Z_i \leq t \right\},$$

choose a largest coordinate $Z_i = x > t/m$. Some $j \neq i$ then satisfies $Z_j > (t-x)/(m-1)$. Partition $[t/m, t]$ into intervals, apply [Eq. \(A.15\)](#) on each interval with the smallest corresponding lower bound for Z_j , and let the mesh tend to zero. This gives

$$\begin{aligned} \mathbb{P}\left(\sum_i Z_i > t, \max_i Z_i \leq t\right) &\leq C_m \int_{t/m}^t \frac{dx}{(1+x)^2(1+t-x)} \\ &= O\left(\frac{\log t}{t^2}\right). \end{aligned}$$

Combining the two bounds proves the proposition. $\square$

Proof of [Theorem A.3](#). If $m = 1$, the conclusion is the exact one-dimensional marginal expansion, so suppose $m \geq 2$. Let $A_i = \{Z_i > t\}$. Exact Pareto and Half-Cauchy margins give, respectively,

$$\mathbb{P}(A_i) = \frac{c_X w_i}{t} + O(t^{-2}).$$

The density bound gives

$$\mathbb{P}(A_i \cap A_j) = O(t^{-2}),$$

so fixed-dimensional Bonferroni bounds yield

$$\mathbb{P}\left(\bigcup_i A_i\right) = \frac{c_X}{t} + O(t^{-2}). \quad (\text{D.1})$$

It remains to bound the event

$$R_t = \left\{ \sum_i Z_i > t, \max_i Z_i \leq t \right\}.$$

On R_t , choose a largest coordinate $Z_i = x$. Then $x > t/m$, and some $j \neq i$ satisfies $Z_j > (t-x)/(m-1)$. Hence

$$\begin{aligned} \mathbb{P}(R_t) &\leq \sum_{i \neq j} \int_{t/m}^t \int_{(t-x)/(m-1)}^\infty g_{ij}(x, y) dy dx \\ &\leq C_m \int_{t/m}^t \frac{dx}{(1+x)^2(1+t-x)} = O\left(\frac{\log t}{t^2}\right). \end{aligned}$$

Combining this bound with [Eq. \(D.1\)](#) proves the result. $\square$

Proof of [Corollary A.4](#). Under monotone marginal transformation, the joint score density is the copula density multiplied by the two marginal score densities. Both a weighted unit-Pareto density and a weighted Half-Cauchy density are bounded by $C_i(1+x)^{-2}$. Therefore [Eq. \(A.18\)](#) implies [Eq. \(A.17\)](#). $\square$

Proof of [Proposition A.5](#). The spectral calculation in [Eq. \(2.12\)](#) gives $\mu_X(A_w) = c_X$. Dividing [Eq. \(A.20\)](#) by t proves [Eq. \(A.21\)](#). $\square$

D.4 Face calculus for higher-order aggregation

Proof of Lemma 5.1. Eq. (5.2) is the Möbius inversion identity on the subset lattice, applied to $g_{[m]}$. $\square$

Proof of Theorem 5.2. Because ν_I is Radon, one may choose $\varepsilon_n \downarrow 0$ and $M_n \uparrow \infty$ so that $\nu_I(\partial K_{\varepsilon_n, M_n}) = 0$: on each fixed compact annulus, the restricted coordinate projections are finite measures and therefore have at most countably many atoms. On each such rectangle, $G_I \mathbf{1}_{K_{\varepsilon_n, M_n}}$ is bounded, compactly supported, and ν_I -almost-everywhere continuous, so vague convergence and the portmanteau theorem give convergence of its integral. Condition Eq. (5.4) removes the truncation uniformly. Summing the finite identity in Lemma 5.1, with Eq. (5.6) for omitted faces, proves the last claim. $\square$

Proof of Proposition 5.3. Put $z_i = w_i x_i$ and $z_j = w_j x_j$. The pair increment equals +1 on

$$0 < z_i, z_j < 1, \quad z_i + z_j > 1,$$

equals -1 on $z_i, z_j > 1$, and vanishes elsewhere. The negative region has finite mass. Near $z_i = 0$, the positive region confines z_j to an interval of length $O(z_i)$, leaving an $O(z_i^{-a_i})$ integrand; this is integrable because $a_i < 1$, and the symmetric argument handles the other axis. Thus the integral is absolutely convergent. Successive integration using Euler's beta identity, continued at $a_i + a_j = 1$, gives Eq. (5.8). $\square$

Proof of Lemma 5.4. If $\Gamma_k(\mathbf{z}) \neq 0$, then $\sum_j z_j > 1$, so at least one coordinate is at least $1/k$. Partition the integration region according to the coordinates exceeding $1/k$.

On a region having at least two such coordinates, integrate those coordinates over $[1/k, \infty)$ and use $|\Gamma_k| \leq 2^k$. Their inverse-square integrals are finite, while each of the remaining $k-2$ coordinates contributes at most $\int_{\varepsilon}^{\infty} z^{-2} dz = \varepsilon^{-1}$. This part is therefore $O\{\varepsilon^{-(k-2)}\}$.

It remains to consider a region on which exactly one coordinate, say z_r , is at least $1/k$. All subset sums not containing r are then strictly below one. With $h(s) = \mathbf{1}\{s > 1\}$,

$$\Gamma_k(\mathbf{z}) = \prod_{j \neq r} \Delta_{z_j} h(z_r), \quad \Delta_a f(s) = f(s+a) - f(s).$$

Commutativity of the difference operators, together with

$$\|\Delta_a f\|_{L^1(\mathbb{R})} \leq 2\|f\|_{L^1(\mathbb{R})}, \quad \|\Delta_a h\|_{L^1(\mathbb{R})} = a,$$

gives

$$\int_{\mathbb{R}} |\Gamma_k(\mathbf{z})| dz_r \leq 2^{k-2} \min_{j \neq r} z_j.$$

Since z_r^{-2} is bounded on $[1/k, \infty)$, the remaining integral is bounded by a constant multiple of

$$\int_{[\varepsilon, 1/k]^{k-1}} \min_j z_j \prod_j z_j^{-2} dz.$$

Writing $n = k-1$ and partitioning according to the smallest coordinate, this is bounded by

$$n \int_{\varepsilon}^{1/k} r^{-1} \left(\int_r^{1/k} z^{-2} dz \right)^{n-1} dr.$$

It is $O\{\log(1/\varepsilon)\}$ when $n = 1$ and $O\{\varepsilon^{-(n-1)}\} = O\{\varepsilon^{-(k-2)}\}$ when $n \geq 2$. Combining the regions proves Eq. (5.9). $\square$

Proof of Theorem 5.5. Apply the exact face decomposition in Lemma 5.1. The singleton terms are exact:

$$\sum_i \mathbb{P}(w_i Y_i > t) = \frac{\sum_i w_i}{t} = \frac{1}{t}.$$

For a pair $i < j$, write

$$\gamma(z_1, z_2) = \mathbf{1}\{z_1 + z_2 > 1\} - \mathbf{1}\{z_1 > 1\} - \mathbf{1}\{z_2 > 1\}.$$

The change of variables $z_1 = w_i Y_i / t$ and $z_2 = w_j Y_j / t$ gives

$$\begin{aligned} \mathbb{E} G_{ij}(Y_i/t, Y_j/t) &= \frac{w_i w_j}{t^2} \int_{w_i/t}^{\infty} \int_{w_j/t}^{\infty} \gamma(z_1, z_2) \\ &\quad \times c_{ij}\left(\frac{w_i}{t z_1}, \frac{w_j}{t z_2}\right) z_1^{-2} z_2^{-2} dz_1 dz_2. \end{aligned} \quad (\text{D.2})$$

If the density in Eq. (D.2) is replaced by one, the exact independent calculation gives

$$\frac{w_i w_j}{t^2} \log \frac{(t - w_i)(t - w_j)}{w_i w_j} = 2w_i w_j \frac{\log t}{t^2} + O(t^{-2}).$$

Replacing one by λ_{ij} therefore produces the proposed logarithmic coefficient.

To control the replacement error, split the positive cooperative region of γ into a central part and its two edge strips. The central part and the negative double-exceedance region give $O(t^{-2})$. On an edge, say $z_2 \downarrow 0$ and $z_1 \in (1 - z_2, 1]$, integration in z_1 supplies a factor proportional to z_2 . The substitution $s = w_j/(t z_2)$ then gives

$$\begin{aligned} &|\mathbb{E} G_{ij}(Y_i/t, Y_j/t) - \lambda_{ij} \mathbb{E}_{\Pi} G_{ij}(Y_i/t, Y_j/t)| \\ &\leq \frac{K_{ij}}{t^2} \left\{ 1 + \int_{K_{ij}/t}^{u_0} \frac{\omega_{ij}(s)}{s} ds \right\}. \end{aligned} \quad (\text{D.3})$$

The lower-strip bound controls the portion on which the second copula argument exceeds u_0 . Since $\omega_{ij}(s) \rightarrow 0$, the integral in Eq. (D.3) is $o(\log t)$ by logarithmic Cesàro averaging. Under Eq. (5.14) it is bounded. This proves the required pair expansion.

Now let I have cardinality $k \geq 3$. On the support of G_I , at least one $w_i Y_i / t$ is at least $1/k$. Hence at least one corresponding copula coordinate is $O(t^{-1})$, so Eq. (5.10) applies for all sufficiently large t . Changing variables $z_i = w_i Y_i / t$ and applying Lemma 5.4 gives

$$\mathbb{E}|G_I(Y_I/t)| \leq K_I t^{-k} \left(\frac{t}{\min_{i \in I} w_i} \right)^{k-2} = O(t^{-2}).$$

There are only finitely many faces. Summing the singleton, pair, and higher-face contributions proves Eqs. (5.13) and (5.15).

Finally, Proposition 2.1 gives a deterministic constant K such that

$$\left| \sum_i w_i H_i - c \sum_i w_i Y_i \right| \leq K.$$

Sandwich the Half-Cauchy tail between the Pareto-sum tails evaluated at $(t-K)/c$ and $(t+K)/c$. Substitution in Eq. (5.13) gives Eq. (5.16); under Eq. (5.14), a bounded threshold displacement changes the tail by only $O(t^{-2})$. $\square$

Proof of Corollary 5.6. Under independence, $\lambda_{ij} = 1$. Subtract Eqs. (4.9) and (4.11) from the corresponding expansions in Eqs. (5.13) and (5.16), and apply Theorem 4.1 with $\kappa = 1$ and $L(t) = \log t$. $\square$

Proof of Theorem A.8. By Eq. (A.33) and independence,

$$\mathbb{P}\{S(Q, \Theta) > t\} = \mathbb{E}\{\bar{F}_Q(q_t)\} + o(t^{-1-\delta}).$$

Insert Eq. (A.31). Condition Eq. (A.34) supplies the leading term and every score-map term at exponent δ . Conditions Eqs. (A.35) and (A.36) show that the radial term contributes $-ab\mathbb{E}(A^{1+\delta})t^{-1-\delta}$ when $\rho = \delta$ and is $o(t^{-1-\delta})$ when $\rho > \delta$. This is Eqs. (A.37) and (A.38). $\square$

Proof of Corollary A.9. In Eq. (A.32), take the relative coefficients B/A at exponent κ and C/A at exponent one, and apply Theorem A.8. Under Eq. (A.42), omit the identically zero exponent-one coefficient and apply the same theorem with the sole score exponent κ . $\square$

Proof of Corollary A.10. The relative score coefficient at exponent κ is B/A and that at exponent two is $-dD/(cA)$. Apply Theorem A.8 with leading angular term cA . $\square$

D.5 The second-order landscape inside index-one MRV

Proof of Proposition A.6. For $tx \geq 1$, exact Pareto margins give

$$t\mathbb{P}(Y_i > tx) = x^{-1} = \mu\{y : y_i > x\}.$$

Substitution in Eq. (A.26) proves Eq. (A.27). If ν_2 is axis-supported, the upper tails in Eq. (A.27) determine its restriction to each axis and make every such restriction zero. The Half-Cauchy assertion follows by expanding $c \arctan\{(tx)^{-1}\}$. $\square$

Proof of Proposition A.7. Fix $x_2 = y$ small. In the positive cooperative part of G_{12} , x_1 lies between $(1 - w_2y)/w_1$ and $1/w_1$, an interval of width proportional to y on which x_1^{-a-1} is bounded above and below. After integrating in x_1 , the boundary contribution is therefore comparable to

$$\int_{\varepsilon}^1 y y^{-a-1} dy = \int_{\varepsilon}^1 y^{-a} dy.$$

This is bounded, logarithmic, or polynomial according as $a < 1$, $a = 1$, or $a > 1$. Symmetry handles the other axis, and all remaining regions are integrable for $a > 0$. $\square$

Proof of Proposition 6.1. Along direction s , all inequalities $rs_i > 1, i \in I$, hold exactly when $r > 1/\min_{i \in I} s_i$. Integrating $r^{-2}dr$ gives Eq. (6.11). The integrand is strictly positive S -almost everywhere, so its integral is positive. MRV then proves Eq. (6.12). $\square$

Proof of Lemma 6.2. Substitute $r = q^{1/\alpha}$ in Eq. (6.13). $\square$

Proof of Lemma A.11. For an edge $e = \{i, j\}$, let k be the omitted vertex and write

$$b_e = w_i + w_j = 1 - w_k, \quad d_e = w_k.$$

Ties among the edge shocks have probability zero. Conditional on $R_e = x$ being the largest edge shock, the two coordinates incident to e equal x , while the remaining coordinate equals $N_e = \max(R_f : f \neq e)$. Consequently, $G_{\mathbf{w}} = b_e x + d_e N_e$. Since $G_{\mathbf{w}} \leq x$ on this event, an exceedance requires $x > t$. Write $h_e(x) = (t - b_e x)/d_e$ and extend F_R by zero below its support. The contribution from the event that R_e is largest is exactly

$$\begin{aligned} P_e(t) &= \int_t^{t/b_e} f_R(x) \{F_R(x)^2 - F_R(h_e(x))^2\} dx \\ &\quad + \int_{t/b_e}^{\infty} f_R(x) F_R(x)^2 dx. \end{aligned} \quad (\text{D.4})$$

The second integral is $ab_e/t + O(t^{-2})$. In the first integral, change variables from x to $h = (t - b_e x)/d_e$. The assumptions give

$$1 - F_R(h)^2 = \frac{2a}{h} + O(h^{-2})$$

for large h . Splitting the h -integral at a fixed constant, and using the density expansion uniformly when both arguments are large, yields

$$\frac{d_e}{b_e} \int_0^t f_R\left(\frac{t - d_e h}{b_e}\right) \{1 - F_R(h)^2\} dh = 2a^2 b_e d_e \frac{\log t}{t^2} + O(t^{-2}).$$

The term involving $1 - F_R\{(t - d_e h)/b_e\}^2$ is $O(t^{-2})$. Therefore

$$P_e(t) = \frac{ab_e}{t} + 2a^2 b_e d_e \frac{\log t}{t^2} + O(t^{-2}).$$

Finally,

$$\sum_e b_e = 2, \quad \sum_e b_e d_e = \sum_{k=1}^3 w_k (1 - w_k) = 2\sigma_2(\mathbf{w}),$$

which proves Eq. (A.52). □

Proof of Theorem 6.3. For $e \in \{12, 13, 23\}$ define

$$W_e = 1 - \exp(-2E_e), \quad Q_e = W_e^{-1}.$$

Then

$$U_1 = \min(W_{12}, W_{13}), \quad U_2 = \min(W_{12}, W_{23}), \quad U_3 = \min(W_{13}, W_{23}),$$

and hence

$$Y_1 = \max(Q_{12}, Q_{13}), \quad Y_2 = \max(Q_{12}, Q_{23}), \quad Y_3 = \max(Q_{13}, Q_{23}).$$

The edge-shock tail and density satisfy

$$\mathbb{P}(Q_e > x) = 1 - \sqrt{1 - x^{-1}} = \frac{1}{2x} + \frac{1}{8x^2} + O(x^{-3}), \quad f_Q(x) = \frac{1}{2x^2} + O(x^{-3}). \quad (\text{D.5})$$

At first order only one edge shock can be large. Such a shock produces the direction $(e_i + e_j)/2$, and $t\mathbb{P}(2Q_e > tr) \rightarrow r^{-1}$. Events containing two large edge shocks have order t^{-2} . This proves [Eq. \(6.18\)](#). Applying [Lemma A.11](#) with $a = 1/2$ proves [Eq. \(6.19\)](#).

Similarly, put $K_e = \cot(\pi W_e/2)$. Monotonicity of the cotangent gives

$$H_1 = \max(K_{12}, K_{13}), \quad H_2 = \max(K_{12}, K_{23}), \quad H_3 = \max(K_{13}, K_{23}).$$

Since

$$\mathbb{P}(K_e > x) = 1 - \sqrt{1 - \frac{2}{\pi} \arctan(x^{-1})} = \frac{c}{2x} + \frac{c^2}{8x^2} + O(x^{-3}), \quad f_K(x) = \frac{c}{2x^2} + O(x^{-3}), \quad (\text{D.6})$$

another application of [Lemma A.11](#), now with $a = c/2$, proves [Eq. \(6.20\)](#). $\square$

Proof of [Corollary 6.4](#). Subtract [Eqs. \(4.9\)](#) and [\(4.11\)](#) from the corresponding expansions in [Eqs. \(6.19\)](#) and [\(6.20\)](#). The independent quantile satisfies $q_{\Pi, S, w}(\alpha) \sim c_S/\alpha$, so substitution into [Eq. \(6.21\)](#) gives [Eq. \(6.22\)](#). Direct substitution in the corresponding pair-shock expansion gives [Eq. \(6.23\)](#). $\square$

D.6 Gaussian copulas

Proof of [Theorem 7.1](#). Fix $i \neq j$ and $x, y > 0$. Let

$$a_s = \Phi^{-1}(1 - sx), \quad b_s = \Phi^{-1}(1 - sy).$$

Both thresholds satisfy

$$a_s \sim b_s \sim \sqrt{2 \log(1/s)}.$$

The simultaneous small- p event is

$$\{U_i \leq sx, U_j \leq sy\} = \{Z_i \geq a_s, Z_j \geq b_s\}.$$

On this event, $Z_i + Z_j \geq a_s + b_s$. Since

$$\text{var}(Z_i + Z_j) = 2(1 + \rho_{ij}) \leq 2(1 + |\rho_{ij}|),$$

the univariate Gaussian tail bound gives

$$\begin{aligned} \mathbb{P}(U_i \leq sx, U_j \leq sy) &\leq \overline{\Phi}\left(\frac{a_s + b_s}{\sqrt{2(1 + |\rho_{ij}|)}}\right) \\ &= s^{2/(1 + |\rho_{ij}|) + o(1)} = o(s), \end{aligned} \quad (\text{D.7})$$

because $2/(1 + |\rho_{ij}|) > 1$. Fixed m makes the conclusion uniform over the finitely many pairs. [Propositions 2.1](#) and [3.3](#) proves the result. $\square$

Proof of [Theorem A.13](#). For every $(i, j) \in E_*$, the bivariate normal tail expansion, uniformly when the two thresholds differ by $O(1)$ on the log-probability scale, gives

$$\mathbb{P}(Y_i > tx_i, Y_j > tx_j) \sim K_{\rho_*} t^{-q_{\rho_*}} (\log t)^{-\beta_{\rho_*}} (x_i x_j)^{-a_{\rho_*}}.$$

Thus the pair hidden measure has the product-power density in [Eq. \(5.7\)](#). [Proposition 5.3](#) gives its weighted shell integral as $K_{\rho_*} C(a_{\rho_*})(w_i w_j)^{a_{\rho_*}}$. All other pair increments are covered directly by [Eq. \(A.62\)](#); hence

only pairs in E_* remain at scale r_* . Given that a maximal pair is simultaneously at its Gaussian extreme level, the conditional regression coefficient of a third coordinate is

$$\frac{\rho_{ki} + \rho_{kj}}{1 + \rho_*} \leq \frac{2\rho_*}{1 + \rho_*} < 1.$$

Gaussian concentration, with strictness supplied by positive definiteness, makes comparable triple extremes smaller than the dominant pair scale. It does not, however, control scale-degenerate regions near proper subspaces. Those regions are exactly what the stated uniform-integrability and face-remainder conditions cover. [Theorem 5.2](#) and [Lemma 5.1](#) now prove [Eq. \(A.63\)](#); no passage from a comparable-threshold bound to a noncompact half-space integral is being assumed.

Since $q_{\rho_*} < 2$, the bounded Half-Cauchy/Pareto comparison in [Proposition 2.1](#), applied at thresholds differing by a fixed constant, transfers the regularly varying second term and multiplies its coefficient by $c^{q_{\rho_*}}$. In the two-sided case the two dominant sign quadrants contribute equally, while the marginal threshold is halved; this changes K_ρ to $2^{1-q_\rho} K_\rho$. $\square$

D.7 Common-factor copulas

Proof of [Lemma A.14](#). As $r \rightarrow \infty$,

$$\begin{aligned} \mathbb{P}(R_0 > r) &= \mathbb{P}(W < \nu/r^2) \\ &\sim \frac{\nu^{\nu/2}}{2^{\nu/2}\Gamma(\nu/2 + 1)} r^{-\nu}. \end{aligned}$$

Thus the common radial multiplier R_0 is regularly varying with index ν . The Gaussian direction $\mathbf{G}$ has moments of every order. The multivariate version of Breiman's lemma therefore implies that $R_0\mathbf{G}$ is MRV with index ν ; see Hult and Lindskog [\[6\]](#) or Basrak, Davis and Mikosch [\[1\]](#). $\square$

Proof of [Theorem A.15](#). For the one-sided case, [Eq. \(A.67\)](#) gives

$$\frac{1}{U_i} \sim \frac{(Z_i^+)^{\nu}}{k_{\nu}} \quad \text{along positive extremes,} \tag{D.8}$$

where $z^+ = \max(z, 0)$. Define

$$g_i(\mathbf{z}) = \frac{(z_i^+)^{\nu}}{k_{\nu}}.$$

The map $\mathbf{g}$ is continuous and homogeneous of degree ν :

$$\mathbf{g}(a\mathbf{z}) = a^{\nu} \mathbf{g}(\mathbf{z}), \quad a > 0.$$

Applying the mapping theorem to the index- ν MRV convergence from [Lemma A.14](#), with the radial scaling raised to the power ν , shows that $\mathbf{g}(\mathbf{Z})$ is MRV with index one. The qualification “along positive extremes” is essential: if $z \leq 0$, then $1 \leq \{1 - T_{\nu}(z)\}^{-1} \leq 2$. Thus negative directions contribute only a bounded score and disappear under score normalization. More precisely,

$$r^{-\nu} \{1 - T_{\nu}(rz)\}^{-1} \longrightarrow \frac{(z^+)^{\nu}}{k_{\nu}},$$

locally uniformly for $\mathbf{z}$ in compact subsets of $\mathbb{R}$. The asymptotically homogeneous mapping theorem therefore transfers the same tail measure to $(1/U_1, \dots, 1/U_m)$. [Proposition 2.1](#) then transfers MRV to the Half-Cauchy scores.

For two-sided p -values,

$$\frac{1}{U_i} \sim \frac{|Z_i|^\nu}{2k_\nu}.$$

The continuous degree- ν map $\mathbf{z} \mapsto (|z_1|^\nu, \dots, |z_m|^\nu)/(2k_\nu)$ gives the same result. $\square$

Proof of Theorem 8.1. Besides Eq. (A.69), direct expansion of the univariate t tail gives

$$\bar{T}_\nu(z) = k_\nu z^{-\nu} \{1 - d_\nu z^{-2} + O(z^{-4})\}.$$

Put $Q = R_0^\nu$. Eq. (A.69) becomes

$$\mathbb{P}(Q > q) = a_\nu q^{-1} \{1 - b_\nu q^{-2/\nu} + O(q^{-4/\nu})\}.$$

On a fixed Gaussian direction, substitution of $Z_i = R_0 G_i$ gives the reciprocal expansion

$$S_Y(q, \mathbf{G}) = qA_s + q^{1-2/\nu}B_s + C_s + o(q^{1-2/\nu} + 1),$$

where $C_s = C_-$ in the one-sided case and $C_s = 0$ in the two-sided case. Gaussian truncation verifies Eqs. (A.34) and (A.36); near a coordinate hyperplane the worst integrand is $|G_i|^{\nu-2}$, which is locally integrable exactly when $\nu > 1$.

For the one-sided statistic, apply Corollary A.9 with the bounded term C_- . For the two-sided statistic, no exponent-one angular term is present. On every Gaussian angular truncation its sharper expansion is

$$S_Y(q, \mathbf{G}) = qA_{|\cdot|} + q^{1-2/\nu}B_{|\cdot|} + O(q^{1-4/\nu}),$$

and the same Gaussian bounds remove the truncation at $\delta_0 = 2/\nu$. The zero- C refinement Eq. (A.43) therefore applies even when $2/\nu > 1$. These two applications, followed when necessary by one further analytic term, yield Eqs. (8.7) and (8.8). In particular, the radial correction dominates the bounded angular term for $\nu > 2$, the terms meet at $\nu = 2$, and the angular term dominates for $1 < \nu < 2$.

For the positive Half-Cauchy aggregate T_w , a positive diverging score satisfies $\cot(\pi U/2) = cU^{-1} + O(U)$, whereas a negative one-sided direction tends to zero. Hence the common-scale expansion has leading angular term cA_s , radial correction cB_s , and no C_- term. The first omitted Half-Cauchy term is of tail order t^{-3} ; because $1 + 2/\nu < 3$ for $\nu > 1$, Corollary A.10 proves Eq. (8.9). $\square$

Proof of Proposition A.16. Ignore the fixed constants in the definitions of A_s and B_s , and put $x_i = G_i^+$ or $x_i = |G_i|$ as appropriate. At $\nu = 2$, the one-sided zero-power factor in B_+ is $\mathbf{1}\{G_i > 0\}$. Thus, with $S_2 = \sum_i w_i x_i^2$ and $S_0 = \sum_i w_i \mathbf{1}\{G_i > 0\} \leq 1$, one has $S_2 S_0 \leq S_2$; in the absolute-value case the corresponding factor is one almost surely. Also $S_2^2 \leq \sum_i w_i x_i^4$ by Jensen's inequality. For $\nu > 2$, weighted power means give the two pointwise inequalities

$$\begin{aligned} \left(\sum_i w_i x_i^\nu \right)^{2/\nu} \left(\sum_i w_i x_i^{\nu-2} \right) &\leq \sum_i w_i x_i^\nu, \\ \left(\sum_i w_i x_i^\nu \right)^{1+2/\nu} &\leq \sum_i w_i x_i^{\nu+2}. \end{aligned}$$

Their expectations are bounded by univariate Gaussian moments, uniformly in the dimension, correlation matrix, and simplex weights.

For $1 < \nu < 2$,

$$\left(\sum_i w_i x_i^\nu \right)^{2/\nu} \leq \sum_i w_i x_i^2.$$

After expanding the resulting double sum, Gaussian regression gives, uniformly in $|\rho| \leq 1$,

$$\mathbb{E}\{|G_i|^2 |G_j|^{\nu-2}\} = (1 - \rho_{ij}^2) \mathbb{E}|N|^{\nu-2} + \rho_{ij}^2 \mathbb{E}|N|^\nu.$$

This proves the asserted bounds for $D_{\nu, \mathbf{w}}^s$. Finally, $0 \leq C_- \leq 1$ gives $\mathbb{E}(A_+ C_-) \leq \mathbb{E}A_+ = k_\nu^{-1} \mathbb{E}(N^+)^{\nu}$ and completes the proof. $\square$

Proof of Theorem 8.2. For fixed positive exponential directions,

$$\sum_i w_i Y_i = R A_{\mathbf{w}} + R^{1-\theta} B_{\mathbf{w}} + o(R^{1-\theta}).$$

Truncation at $E_i \in [\varepsilon, M]$, followed by exponential moment bounds near zero and infinity, verifies Eqs. (A.34) and (A.36) for every $\theta > 0$. The zero- C refinement Eq. (A.43), with $Q = R$ and $\rho = \kappa = \theta$, gives Eq. (8.14); exact marginal uniformity gives $a_\theta \mathbb{E}A_{\mathbf{w}} = 1$.

For an exponential variable, integration by parts gives $\mathbb{E}f'(E) = \mathbb{E}f(E) - f(0)$. Apply this to $f = A_{\mathbf{w}}^{\theta+1}$ in every coordinate to obtain

$$\mathbb{E}(A_{\mathbf{w}}^\theta B_{\mathbf{w}}) = \frac{1}{\theta + 1} \left\{ m \mathbb{E}A_{\mathbf{w}}^{\theta+1} - \sum_i \mathbb{E}(A_{\mathbf{w}} - w_i E_i^P)^{\theta+1} \right\}.$$

Together with $b_\theta = 1/(1+\theta)$ this proves Eq. (8.15). Finally, if $x_i = w_i E_i^P$ and $A = \sum_i x_i$, then $(1 - x_i/A)^{\theta+1} \leq 1 - x_i/A$. Summing proves nonnegativity, with strict inequality whenever two x_i are positive. $\square$

Proof of Theorem 8.3. Insert Eq. (8.11) into Eq. (8.16). The common-factor statistic has a Clayton correction at score-scale exponent $\kappa = \theta$ and a transform correction $-dR^{-1}Q_{\mathbf{w}}$ at exponent $\kappa = 2$. For $1 < \theta < 2$, and for the cases $\theta \geq 2$, Corollary A.10 gives the asserted result and adds the tied terms at $\theta = 2$. The required inverse exponential moment exists in these regimes. Indeed, weighted Jensen and expansion of the two weighted sums bound

$$\mathbb{E}(A_{\mathbf{w}}^2 Q_{\mathbf{w}}) \leq \max\{\Gamma(1+p), \Gamma(1+2p)\Gamma(1-p)\} < \infty, \quad p = 1/\theta < 1.$$

It remains to justify (i) without this inverse moment. If $0 < \theta < 1$, Proposition 2.1 and the simplex weights give

$$cR_{\mathbf{w}} - c \leq T_{\mathbf{w}} \leq cR_{\mathbf{w}}.$$

The fixed threshold displacement changes the leading c/t tail by $O(t^{-2}) = o(t^{-1-\theta})$. Applying Eq. (8.14) at t/c therefore transfers its $c^{1+\theta} D_{\theta, \mathbf{w}}$ coefficient to $T_{\mathbf{w}}$.

At $\theta = 1$, use the sharper scalar bound

$$0 \leq cy - \cot\left(\frac{\pi}{2y}\right) \leq \frac{C}{y}, \quad y \geq 1, \quad (\text{D.9})$$

which follows from Eq. (8.16) and continuity on compact intervals. Here $Y_i = 1 + RE_i$, $R = V^{-1}$ with V standard exponential, and $A = \sum_i w_i E_i$. Put $D_E(r) = \sum_i w_i / (1 + rE_i)$. Conditional on the exponentials,

the reciprocal threshold is $r_t = (t/c - 1)/A$. On the discrepancy event $\{cR_w > t \geq T_w\}$, [Eq. \(D.9\)](#) and monotonicity of D_E imply

$$0 < R - r_t \leq \frac{CD_E(r_t)}{cA}.$$

Since the density of R is $e^{-1/r}r^{-2} \leq r^{-2}$, the probability lost when replacing cR_w by T_w is at most a constant times

$$\mathbb{E} \left\{ \frac{1}{Ar_t^2} \sum_i \frac{w_i}{1 + r_t E_i} \right\} = O \left(t^{-3} \sum_i w_i \mathbb{E} \frac{A^2}{E_i + A/t} \right) = O(t^{-3} \log t).$$

To justify the final bound, fix an active i and write $B_i = \sum_{j \neq i} w_j E_j$, so $A = w_i E_i + B_i$. Conditional on $B_i = b$, for all sufficiently large t ,

$$\frac{A^2}{E_i + A/t} \leq C_w \left\{ E_i + \frac{b^2}{E_i + b/t} \right\}.$$

For $0 < b \leq t$,

$$\int_0^\infty \frac{e^{-x}}{x + b/t} dx \leq C \{1 + \log(t/b)\},$$

whereas for $b > t$ the integral is at most t/b . Exponential tails and $\mathbb{E}[B_i^2 \{1 + |\log B_i|\}] < \infty$ therefore give the required $O(\log t)$ expectation. If only one weight is active, $B_i = 0$ and the expectation is $O(1)$. Thus the probability loss is $o(t^{-2})$, so the $\theta = 1$ reciprocal coefficient also transfers, proving (i) throughout $0 < \theta < 2$.

The one-coordinate identity follows from gamma moments or directly from the Half-Cauchy tail. $\square$

Proof of [Proposition A.17](#). The lower bound was proved above. The integration-by-parts identity and $A^r - (A - x)^r \leq rA^{r-1}x$ give $\mathbb{E}(A^\theta B) \leq \mathbb{E}A^{\theta+1}$. Weighted Jensen then gives

$$\mathbb{E}A^{\theta+1} \leq \mathbb{E}E_1^{(\theta+1)/\theta} = \Gamma(2 + 1/\theta).$$

Retaining the negative radial term gives

$$D_{\theta, w} \leq a_\theta(1 - b_\theta)\mathbb{E}A^{\theta+1} \leq a_\theta \frac{\theta}{1 + \theta} \Gamma(2 + 1/\theta) = 1.$$

The final claim is the estimate in the proof of [Theorem 8.3](#). $\square$

D.8 Growing-dimensional results

Proof of [Theorem A.18](#). For every n , the spectral representation and [Eq. \(A.73\)](#) give

$$\mu_n(A_n) = \sum_{i=1}^{m_n} w_{i,n} \mu_n\{\mathbf{x} : x_i > 1\} = c_X.$$

Substitution into [Eq. \(A.74\)](#) proves [Eq. \(A.75\)](#). $\square$

Proof of [Theorem A.19](#). Sandwich the half-space indicator between linear ramps of width η in $\mathbf{w}^\top \mathbf{x}$. Because simplex weights have Euclidean norm at most one, the ramps have Lipschitz constant at most η^{-1} and are supported away from the origin. Homogeneity and the marginal normalization give $\mu_n\{\mathbf{w}^\top \mathbf{x} > s\} = c/s$. The lower ramp loses at most

$$c - \frac{c}{1 + \eta} = \frac{c\eta}{1 + \eta},$$

whereas the upper ramp gains at most

$$\frac{c}{1-\eta} - c = \frac{c\eta}{1-\eta}.$$

Both one-sided errors are bounded by $c\eta/(1-\eta)$, which proves [Eq. \(A.77\)](#). The larger mass of the full two-sided shell is not needed. $\square$

Proof of [Theorem A.20](#). Condition on the angle and substitute the radial tail formula at $t/L_{n,w}$. Since $0 \leq L_{n,w} \leq 1$ and $t/L_{n,w} \geq t$ on $\{L_{n,w} > 0\}$, the assumed radial expansion is valid simultaneously for all angles once t is sufficiently large. Since $C_n \mathbb{E} L_{n,w} = c$, the leading term is exactly c and the remainder is [Eq. \(A.78\)](#). The conditional version is identical. $\square$

Proof of [Lemma A.21](#). Put $h(q) = q\mathbb{P}(Q > q)$, so $h(q) \rightarrow a$. Fix q_0 large and split at $S_n = t_n/q_0$. On $S_n \leq t_n/q_0$,

$$t_n \mathbb{P}(Q > t_n/S_n \mid S_n) = S_n h(t_n/S_n),$$

with the expression defined as zero when $S_n = 0$. The difference from aS_n is at most $\sup_{q \geq q_0} |h(q) - a|S_n$. On the complementary event, Markov's inequality and the uniform $(1+\delta)$ moment bound give

$$t_n \mathbb{P}(S_n > t_n/q_0) = O(t_n^{-\delta}), \quad \mathbb{E}\{S_n \mathbf{1}(S_n > t_n/q_0)\} = O(t_n^{-\delta}).$$

First let $n \rightarrow \infty$ and then $q_0 \rightarrow \infty$. $\square$

Proof of [Theorem A.22](#). Write $\mathbf{Z}_n = R_0 \mathbf{G}_n$ as in [Eq. \(A.66\)](#). Scalar regular variation of the exact t_ν score gives, for every $\varepsilon > 0$, upper and lower envelopes of the form

$$(1-\varepsilon)R_0^\nu S_n - C_\varepsilon \leq \sum_i w_{i,n} Y_{i,n} \leq (1+\varepsilon)R_0^\nu S_n + C_\varepsilon,$$

where S_n is A_+ or $A_{|\cdot|}$ from [Eqs. \(8.3\)](#) and [\(8.5\)](#). The constants do not depend on m_n , because the weights sum to one. Also $a_\nu \mathbb{E} S_n = 1$ and, for some fixed $\delta > 0$,

$$\sup_n \mathbb{E} S_n^{1+\delta} < \infty$$

by weighted Jensen and the univariate Gaussian moments. Applying [Lemma A.21](#) to the fixed regularly varying factor R_0^ν , and then letting the envelope error tend to zero, proves [Eq. \(A.80\)](#). The deterministic uniform bound between the weighted Half-Cauchy and scaled reciprocal sums proves [Eq. \(A.81\)](#). $\square$

Proof of [Theorem A.23](#). In the frailty representation [Eq. \(8.11\)](#), put $S_n = \sum_i w_{i,n} E_i^{1/\theta}$. For $1/\theta \leq 1$, subadditivity gives

$$RS_n \leq \sum_i w_{i,n} Y_i \leq RS_n + 1.$$

For $p = 1/\theta > 1$, use the elementary convexity bound

$$(a+b)^p \leq (1+\eta)^{p-1} a^p + (1+\eta^{-1})^{p-1} b^p, \quad a, b \geq 0.$$

Choosing η so that $(1+\eta)^{p-1} = 1+\varepsilon$ gives the same sandwich with $(1+\varepsilon)RS_n + C_{\varepsilon,\theta}$ on the upper side. Furthermore, $a_\theta \mathbb{E} S_n = 1$ and weighted Jensen yields

$$\sup_n \mathbb{E} S_n^{1+\delta} \leq \mathbb{E} E_1^{(1+\delta)/\theta} < \infty$$

for every fixed $\delta > 0$. Uniform Breiman convergence for the fixed index-one factor R , now justified by [Lemma A.21](#), proves the reciprocal result. The weighted Half-Cauchy/Pareto difference is bounded independently of m_n , proving the positive Half-Cauchy result. $\square$

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